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Topological Formulas for Network Functions

by

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CONTENTS

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I. INTRODUCTION

Topological formulas for computing currents and voltages in an electrical network are by no means new. Kirchhoff[15] stated the basic formulas for mesh equations in 1847 and Maxwell[18] gave the basic formulas for the node equations in 1872. Since that time many articles have been written on these rules, both to popularize them and also to give proofs of the rules.[3, 5, 10, 11, 12, 17, 19, 23, 25]*

The objectives of this report are threefold. The first is to give a unified and fairly complete picture of the subject which can be read without referring extensively to other papers. The proofs of the fundamental topological properties are omitted in Chapter II, as these statements are fairly well known and their inclusion would lengthen the report by a disproportionate amount. With this exception the report is complete. The available papers in the literature are sketchy, and they use varying terminology and notation.

The second objective is to present formal and precise proofs of a number of topological formulas which now are supported mainly by heuristic reasoning. In particular, the formulas for transfer functions previously had no correct proofs. The last object is to present a number of new topological formulas for the open and short circuit functions and current and voltage ratio transfer functions of two terminal-pair networks.

The current interest in topological formulas has developed because of the possibilities they present in discovering radically new synthesis procedures for lumped networks (which can be extended to cover all lumped systems).

Chapter II serves primarily to explain notations and terminology and to collect the fundamental results on network topology. No proofs are included in this chapter but references are made to papers where they may be found. A general knowledge of network analysis and matrix algebra are assumed.

Chapter III deals with one terminal-pair networks. Topological formulas for mesh and node determinants, their inter-relationship, and formulas for driving point functions are the main results of this chapter. All the proofs are included in this and subsequent chapters.

Chapters IV, V, and VI are concerned with the more interesting two terminal-pair networks. Transfer admittances and short circuit functions are expressed in terms of element admittances. These formulas are extended to include current and voltage ratio transfer functions and the computation of T and π network equivalents. Similar formulas are derived from mesh equations for the open circuit functions and the current and voltage ratio transfer functions of two terminal-pair networks. These are expressed in terms of impedances of network elements.

The possibilities in network synthesis which are offered by topological formulas are discussed and the relevant unsolved problems are stated in Chapter VII.

The authors wish to acknowledge with gratitude the help given by Mr. S. Hakimi, who read the report critically and prepared the examples.

1. Terms and Symbols

The definitions of terms and symbols used in this report are given below.

Definitions of Terms

Element — line segment together with its end points.

Vertex — end point of an element.

Incidence — A vertex and an element are incident to each other if the vertex is an end point of the element.

Graph — finite collection of elements such that no two elements have a point in common that is not a vertex.

Oriented Element — an element together with an ordering of its end points.

Tree — connected subgraph of a connected graph containing all the vertices of the graph, but containing no circuits.

* Superscripts in parentheses refer to corresponding entries in the references.

Branch — element of a tree.

Chord — element of the complement of a tree.

Circuit — connected graph or subgraph in which every vertex has two elements incident to it.

2-Tree — pair of unconnected circuitless subgraphs (each subgraph being connected) which together include all the vertices of the graph. Either or both may consist of isolated vertices.

3-Tree — set of three unconnected subgraphs (each subgraph being connected) which together include all the vertices of the graph but contain no circuits.

Fundamental Circuits — for a given tree T of a connected graph G, the set of fundamental circuits is the set of $e - v + 1$ circuits, each circuit containing exactly one chord of the tree T.

One Terminal-Pair Network — a network with a pair of vertices designated as "input vertices."

Two Terminal-Pair Network — a network with one pair of vertices designated as "input vertices" and one pair of vertices designated as "output vertices."

Driving Point Impedance and Admittance — for a one terminal-pair network containing no drivers, the driving point impedance is

$$Z_d(s) = V(s)/I(s)$$

where $V(s)$ is the Laplace transform of the input voltage, $I(s)$ is the Laplace transform of the input current and all the initial voltages and currents in the network are zero. The reciprocal of $Z_d(s)$ is the driving point admittance.

Transfer Impedance and Admittance — transfer admittance of a two terminal-pair network is the ratio

$$Y_{12}(s) = I_1(s)/V_2(s)$$

where $I_1(s)$ is the Laplace transform of the input current, $V_2(s)$ is the Laplace transform of the output voltage and all the initial currents and voltages are zero. The reciprocal of the function $Y_{12}(s)$ is the transfer impedance $Z_{12}(s)$.

Voltage Ratio Transfer Function — for a two terminal-pair network with input vertices $(1, 1')$ and output vertices $(2, 2')$, the transfer voltage ratios are

$$\mu_{12}(s) = V_2(s)/V_1(s)$$

with end 2 open-circuited and all initial conditions zero; and

$$\mu_{21}(s) = V_1(s)/V_2(s)$$

with end 1 open-circuited and all initial conditions zero.

Current Ratio Transfer Function — for a two terminal-pair network with input vertices $(1, 1')$ and output vertices $(2, 2')$ the current ratio transfer functions are:

$$\alpha_{12}(s) = I_2(s)/I_1(s)$$

with end 2 short circuited and all initial conditions zero; and

$$\alpha_{21}(s) = I_1(s)/I_2(s)$$

with end 1 short circuited and all initial conditions zero.

Definitions of Symbols:

A_a = incidence matrix of the graph including all vertices. Rows correspond to vertices and columns to elements.

A = incidence matrix with dependent rows deleted.

A_{-i} = matrix A with row i deleted.

B_a = circuit matrix with one row for each circuit in the graph.

B = circuit matrix with the dependent rows deleted.

B_{-i} = matrix B with row i deleted.

C = capacitance.

D = reciprocal of capacitance. $D=1/C$

D_e = reciprocal capacitance matrix for elements.

E = voltage of a driver.

E_m = column matrix of mesh voltage drivers.

G = graph.

G_s = subgraph.

I = current.

I_{2TP} = two terminal-pair current matrix. $I_{2TP} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$

J = current driver.

L = inductance.

L_e = element inductance matrix including mutual inductances.
M = mutual inductance.
M_{ij} = minor of a determinant obtained by deleting row i and column j.
R = resistance.
R_e = element resistance matrix.
T = tree.
T_2 = 2-tree.
$T_{2_{ij,k}}$ = a 2-tree with vertices i and j in one connected part and vertex k in the other.
T_3 = a 3-tree.
$T_{3_{i,jk,m}}$ = a 3-tree with vertex i in one part, vertices j, k in another and vertex m in the third part.
$U(Y)$ = sum of 3-tree admittance products.
$V(Y)$ = sum of tree admittance products.
$W(Y)$ = sum of 2-tree admittance products.
$V(Z)$ = sum of tree impedance products.
$W(Z)$ = sum of 2-tree impedance products.
V_{ij} = voltage between nodes i and j with reference + at i.
V_e = column matrix of element voltages.
V_n = column matrix of node voltages.
V_{2TP} = two terminal-pair voltage matrix. $V_{2TP} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$
Y = admittance.
Y_e = element admittance matrix.
Y_n = node admittance matrix.
Y_{sc} = short circuit admittance matrix of two terminal-pair network.
Z = impedance.
Z_m = mesh impedance matrix.
Z_e = element impedance matrix.
Z_{oc} = open circuit impedance matrix of a two terminal-pair network.
a_{ij} = typical element of incidence matrix.
b_{ij} = typical element of the circuit matrix.
b_i = branch i of a tree.
c_i = chord i of a tree.
e = element of a graph.
g = conductance.
y_i = admittance of element i.
z_i = impedance of element i.
α = current ratio transfer function.
Γ = reciprocal inductance.
Δ = determinant.
Δ_m = mesh determinant.
Δ_n = node determinant.
Δ_{ab} = cofactor of an element in the (a, b) position.
$\Delta_{ab,cd}(-1)^{a+b+c+d}$ = determinant obtained by deleting rows a and c, and columns b and d.
μ = voltage ratio transfer function.
Σ = summation symbol.

II. PRELIMINARY RESULTS

2. The Graph of a Network

If a person were to forget about currents, voltages, resistances, inductances and capacitances, there would still be one aspect of the electrical network remaining — its geometry. The study of this geometry is known as topology. In representing this geometry it is usual to show all network elements as line segments (instead of using the symbols of R, L, C). For example, the purely geometrical representation of the network of Fig. 1 is shown on Fig. 2.

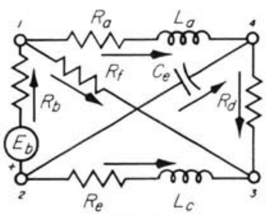


Fig. 1.

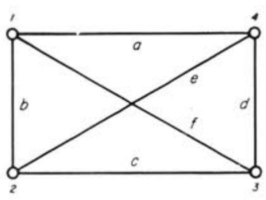


Fig. 2

A configuration of interconnected line segments, such as Fig. 2, is known as a graph and the line segments of the graph are elements. In electrical network theory the line segments of the graph are considered as oriented. This orientation is to take care of current and voltage references used in network theory. The orientation is shown by means of an arrowhead placed on each element. (Abstractly an element is oriented by ordering its end points.) The arrowhead is in the same direction as the current reference or "assumed positive direction of current." The reference positive polarity of the voltage is considered to be at the tail of the arrow orienting the line segment. A graph in which every element is so oriented, is known as an oriented graph. For example the oriented graph corresponding to the network of Fig. 1 is shown in Fig. 3.

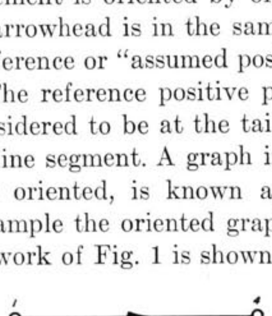


Fig. 3.

3. Trees, Chords, and Branches

Topological formulas are all stated in terms of the basic concept of a tree. A tree of a graph is a connected subgraph which contains all the nodes but no circuits. (The standard mathematical definition does not demand that all the nodes be included, but this "complete tree" is the more useful notion in electrical network theory.) For example, the graph of Fig. 3 has sixteen trees, four similar to each of the trees in Figs. 4a to 4d.

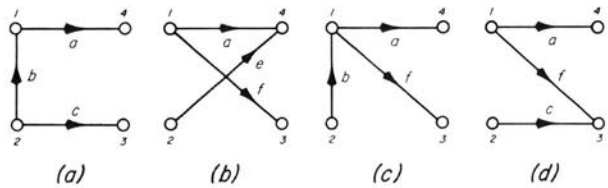


Fig. 4.

Despite the simplicity of this concept, the tree is a very fundamental notion because of its close relationship to the structure of network matrices and, therefore, to the topological formulas. The elements of a tree are known as branches while the elements of the network which are not in the chosen tree are called chords. For example, in the tree in Fig. 4a, the branches are a, b, and c, and the chords are d, e, and f. The tree must be designated before the terms chords and branches can be used. The complement of a tree (the set of chords) is sometimes called a co-tree.(19)

4. Incidence Matrix

It is quite obvious that the graph is completely specified by giving the end points and orientation of every element of the graph. Such a specification is made most conveniently by means of a matrix, known as the incidence matrix. The rows of the incidence matrix correspond to the vertices or nodes of the graph and the columns correspond to the elements of the graph. The formal definition is as follows:

The incidence matrix $A = [a_{ij}]$ is of order (v, e) (i.e., it has v rows and e columns), where v is the number of vertices and e is the number of elements, and

$a_{ij} = 1$ if element j is at vertex i and oriented away from vertex i;

$a_{ij} = -1$ if element j is at vertex i and oriented towards vertex i;

$a_{ij} = 0$ if element j is not at vertex i.

For example, the incidence matrix of Fig. 3 is:

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array} \qquad (1)$$

The subscript a on A_a indicates that all the vertices have been included. Three fundamental properties of the matrix A_a are:

1. The rank of the matrix A_a for a connected network is $v-1$, where v is the number of vertices.
2. Every non-singular submatrix of A_a of order $v-1$ corresponds to a tree; conversely, every square submatrix of order $v-1$ corresponding to a tree is non-singular.
3. The determinant of any square submatrix of A_a is 1, -1, or 0.

(Proofs of these statements may be found elsewhere.)[5, 21, 22]

For example, the submatrix consisting of the columns a-b-c of A_a above, with rows 1, 2, 3, is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad (2)$$

which is non-singular. The elements a-b-c constitute a tree. Corresponding to the tree b-c-d, the submatrix of rows 1, 3, 4 is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \qquad (3)$$

which is non-singular. On the other hand, elements c-d-e do not constitute a tree. The submatrix of A_a with columns c-d-e is

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad (4)$$

which has no non-singular submatrix of order 3.

These three properties constitute the basis of all topological formulas for admittance functions.

Since the rank of A_a of a connected graph is one less than the number of rows, it is usual to drop one row (any row may be dropped). The remaining matrix of order $(v-1, e)$ and rank $v-1$ is symbolized as A. The matrix A is the coefficient matrix of Kirchoff's current law equations, which may be written as[21]

$$A\, I_e(t) = 0$$

where $I_e(t)$ is a column matrix of current functions of elements.

5. Circuit Matrix

A circuit of a graph is a simple closed path. "Loop" and "mesh" are two other words commonly used for a circuit. In network theory the circuit is oriented, which is shown by an arrow. The graph of Fig. 3 is redrawn in Fig. 5, showing five of the oriented circuits.

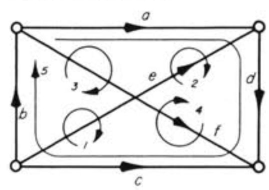


Fig. 5.

There are two other circuits, consisting of elements a-f-c-e and b-e-d-f, respectively, which are not shown. The relationship between the elements and circuits is expressed most conveniently by means of a matrix. This matrix, known as the circuit matrix, has one row for every possible circuit and one column for each element. The circuit matrix

$$B_a = [b_{ij}]$$

is defined as

$b_{ij} = 1$ if element number j is in circuit number i and the orientations of the element and the circuit coincide;

$b_{ij} = -1$ if element number j is in circuit number i and the orientations of the element and the circuit are opposite;

$b_{ij} = 0$ if element number j is not in circuit number i.

For the circuits of Fig. 5, the circuit matrix is given by:

$$B_a = \begin{array}{c} \begin{array}{ccccccc} & a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 & -1 \\ 3 & 1 & 1 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 & 1 & 1 & 0 \\ 5 & 1 & 1 & -1 & 1 & 0 & 0 \\ 6 & 1 & 0 & 1 & 0 & -1 & -1 \\ 7 & 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array} \qquad (5)$$

The following properties of the circuit matrix are fundamental in the study of electrical networks:

1. The rank of the circuit matrix of a connected graph is $e-v+1$ where e is the number of elements and v is the number of vertices;

2. Every non-singular square submatrix of B_a of order $e-v+1$ corresponds to a chord set of some tree.

The circuit matrix differs from the incidence matrix in that while any $(v-1)$ rows of A_a constitute a submatrix of rank $v-1$, not every submatrix of B_a of $e-v+1$ rows has a rank $e-v+1$. For example, the submatrix of rows 2, 3, 7 of the B_a in Eq. 5 has a rank of only 2. The rules for selecting appropriate sets of $e-v+1$ circuits are well known[21] and will not be given here. A submatrix of B_a of $e-v+1$ rows and rank $e-v+1$ is denoted as B. In such a matrix B the converse of property 2 is also true. Every submatrix of B corresponding to a chord set is non-singular. The proofs are found elsewhere.[21]

A useful case of a set of $e-v+1$ circuits is known as the fundamental set of circuits. This set is obtained in the following fashion. A tree of the graph is chosen, then circuits are formed, each consisting of one chord and the tree path between the vertices of the chord. The circuit orientation is chosen to coincide with the chord orientation. The matrix of these fundamental circuits (when suitably arranged) contains a unit matrix of order $e-v+1$.

6. Mesh and Node Systems of Equations

The fundamental equations of electrical network theory, in matrix notation are:

$$A\,I_e(t) = 0 \qquad \text{(Kirchhoff's Current Law)} \quad (6)$$

$$B\,V_e(t) = 0 \qquad \text{(Kirchhoff's Voltage Law)} \quad (7)$$

$$V_e(t) = L_e \frac{d}{dt} I_e(t) + R_e I_e(t) + D_e \int_0^t I_e(x)\,dx + V_c(0) + E_e(t). \quad (8)$$

Kirchhoff's current and voltage laws are solvable respectively by

$$I_e(t) = B' I_m(t) \qquad \text{(Mesh Transformation)} \quad (9)$$

$$V_e(t) = A' V_n(t) \qquad \text{(Node Transformation)} \quad (10)$$

where the functions $I_m(t)$ and $V_n(t)$ are known respectively as mesh currents and node voltages. The prime indicates the transposed matrix. The validity of the mesh and node transformations depends upon the interrelation between matrices A and B:

$$AB' = 0; \qquad BA' = 0. \quad (11)$$

The formal proof may be found elsewhere.[21]

The mesh and node equations are derived here using Laplace transforms. Taking transforms of the fundamental equations

$$A\,I_e(s) = 0 \qquad \text{(Kirchhoff's Current Law)} \quad (12)$$

$$B\,V_e(s) = 0 \qquad \text{(Kirchhoff's Voltage Law)} \quad (13)$$

$$\begin{aligned} V_e(s) &= E_e(s) + (sL_e + R_e + \frac{1}{s} D_e)\,I_e(s) \\ &\quad - L_e I_e(0+) + \frac{1}{s} V_c(0+) \\ &= E_e(s) + Z_e(s)\,I_e(s) \\ &\quad - L_e I_e(0+) + \frac{1}{s} V_c(0+) \end{aligned} \quad (14)$$

$$I_e(s) = B' I_m(s) \quad \text{(Mesh Transformation)} \quad (15)$$

$$V_e(s) = A' V_n(s) \quad \text{(Node Transformation)} \quad (16)$$

Substituting Eq. 14 for $V_e(s)$ in Kirchhoff's voltage law and using the mesh transformation (15) the mesh system of equations is:

$$B\,Z_e\,B'\,I_m(s) = -B\,E_e(s) + B\,L_e\,I_e(0+) - \frac{1}{s} V_c(0+). \quad (17)$$

Solving Eq. 14 for $I_e(s)$ and using $Y_e(s) = Z_e^{-1}(s)$

$$I_e(s) = Y_e(s) \left\{ V_e(s) - E_e(s) + L_e\,I_e(0+) - \frac{1}{s} V_c(0+) \right\}. \quad (18)$$

Substituting this expression for $I_e(s)$ in Kirchhoff's current law and using the node transformation the node system of equations is:

$$A\,Y_e A'\,V_n(s) = A\,Y_e(s)\,E_e(s) - A\,Y_e(s)\,L_e\,I_e(0+) + \frac{1}{s} A\,Y_e(s)\,V_c(0+). \quad (19)$$

The network functions are all defined for the conditions[27]

$$I_L(0+) = V_c(0+) = 0.$$

Using these initial conditions the mesh and node equations may be written in their final useful forms as:

$$Z_m(s)\,I_m(s) = E_m(s) \quad (20)$$

and

$$Y_n(s)\,V_n(s) = J_n(s) \quad (21)$$

where

$$Z_m(s) = B\,Z_e(s)\,B'; \quad E_m(s) = -B\,E_e(s);$$
$$Y_n = A\,Y_e\,A'; \quad J_n = Y_e\,E_e. \tag{22}$$

In this report only networks in which there are no mutual inductances are considered. For such networks the inductance matrix L_e is diagonal. The resistance and reciprocal capacitance matrices R_e and D_e are always diagonal. Therefore, for such networks, Z_e and Y_e are both diagonal matrices.

The topological formulas discussed in this report are for the determinants and cofactors of the mesh and node systems of Eqs. 20 and 21.

Example:

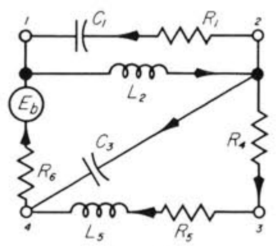


Fig. 6.

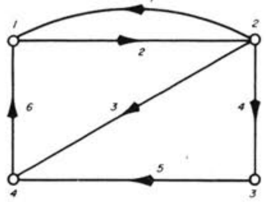


Fig. 7.

The system of equations

$$A\,I_e(s) = 0 \tag{23}$$

then becomes

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \\ I_6(s) \end{bmatrix} = 0 \tag{24}$$

The system of equations

$$B\,V_e(s) = 0 \tag{25}$$

becomes, on choosing fundamental circuits for the tree 4-5-6,

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \\ V_6(s) \end{bmatrix} = 0 \tag{26}$$

The system of equations

$$V_e(s) = E_e(s) + \left(sL_e + R_e + \frac{1}{s}D_e\right) I_e(s) - L_e\,I_e(0+) + \frac{1}{s}\,V_e(0+) \tag{27}$$

becomes:

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \\ V_6(s) \end{bmatrix} = \begin{bmatrix} R_1+D_1/s & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_2 & 0 & 0 & sL_{25} & 0 \\ 0 & 0 & D_3/s & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & sL_{52} & 0 & 0 & R_5+sL_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \\ I_6(s) \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & L_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{52} & 0 & 0 & L_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_2(0+) \\ 0 \\ 0 \\ I_5(0+) \\ 0 \end{bmatrix}$$
$$+ \frac{1}{s}\begin{bmatrix} V_1(0+) \\ 0 \\ V_3(0+) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ E_6(s) \end{bmatrix} \tag{28}$$

L_{25} and L_{52} denote the mutual coupling between the elements 2 and 5.

The mesh transformation is

$$I_e(s) = B'I_m(s) \tag{29}$$

which becomes for this network:

$$\begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \\ I_4(s) \\ I_5(s) \\ I_6(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_{m1}(s) \\ I_{m2}(s) \\ I_{m3}(s) \end{bmatrix} \tag{30}$$

The node transformation is

$$V_e(s) = A'V_n(s) \tag{31}$$

which becomes for this network:

$$\begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ V_4(s) \\ V_5(s) \\ V_6(s) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{n1}(s) \\ V_{n2}(s) \\ V_{n3}(s) \end{bmatrix} \tag{32}$$

III. TOPOLOGICAL FORMULAS FOR ONE TERMINAL-PAIR NETWORKS

7. Mesh and Node Determinants

The node determinant Δ_n is given by

$$\Delta_n = \det Y_n(s) = \det (A\ Y_e\ A'). \qquad (33)$$

The Binet-Cauchy theorem[2] is used to compute this determinant. A major determinant or a major of a rectangular matrix is the determinant of a largest square submatrix that can be formed from the matrix. For example, a major of the incidence matrix A is a determinant of order $(v-1)$ selected from A. Then the Binet-Cauchy theorem states that

$$\Delta_n = \det Y_n(s) = \Sigma \text{ Products of corresponding majors of } [A \cdot Y_e(s)] \text{ and } A'. \qquad (34)$$

Corresponding major means the following. If the major selected from $A \cdot Y_e(s)$ consists of columns $c_1, c_2, \ldots, c_{v-1}$, then the corresponding major of A' consists of rows $c_1, c_2, \ldots, c_{v-1}$. The summation is over all such corresponding majors.

Since $Y_e(s)$ is a diagonal matrix, the product $A \cdot Y_e(s)$ is a matrix, the same as A, except that the first column is multiplied by y_1, the second column by y_2, etc. A major of $A \cdot Y_e(s)$ consisting of columns $i_1, i_2, \ldots, i_{v-1}$ therefore has the value

$$y_{i_1} y_{i_2} \ldots y_{i_{v-1}} \text{ (major of } A).$$

By the results of Section 3 (property 2) the major of A is non-zero only if the elements $i_1, i_2, \ldots, i_{v-1}$ constitute a tree. The corresponding major of A' consists of rows $i_1, i_2, \ldots, i_{v-1}$ and so is simply the transpose of the major selected from A. Hence it has the same value and we have:

$$\Delta_n = \Sigma_i (y_{i_1} y_{i_2} \ldots y_{i_{v-1}}) \text{ (major of } A)^2. \qquad (35)$$

By property 3 of Section 3, every non-zero major of A has the value ± 1. Thus Δ_n is simply the sum of the products of the admittances for every set of $(v-1)$ elements constituting a tree of the network. If a tree product is defined to be the product of admittances of the branches of a tree, the following topological formula is derived for the node determinant:

$T\,1$: For a network that contains no mutual inductances, the node determinant is given by

$$\Delta_n = \det Y_n(s) = \Sigma \text{ Tree Products}. \qquad (36)$$

Maxwell[18] originally gave this formula as:

Δ_n is the sum of products of the conductivities taken $v-1$ at a time, omitting all those terms which contain products of the conductivities of branches which form closed circuits.

For the mesh determinant, very similar considerations apply.

$$\Delta_m = \det B \cdot Z_e \cdot B' = \Sigma \text{ Products of corresponding majors of } B \cdot Z_e \text{ and } B'. \qquad (37)$$

For a network that contains no mutual inductances, $Z_e(s)$ is diagonal and so as before,

$$\Delta_m = \Sigma_i z_{i_1} z_{i_2} \ldots z_{i_\mu} \text{ (major of } B)^2 \qquad (38)$$

where $\mu = e - v + 1$.

According to Property 2 of Section 4, the non-zero majors of B are in one-one correspondence with the chord sets of the network. However, such a major does not necessarily have a value ± 1. Okada[19] shows that the value of a non-zero major of B is $\pm 2^i$, i being a non-negative integer, fixed for a given B. Thus if a chord set product is defined to be the product of the impedances of the chords of a tree of the network, the following topological formula is derived for the mesh determinant:

$T\,2$: For a network that contains no mutual inductances,

$$\Delta_m = \det B \cdot Z_e \cdot B' = 2^{2i}\, \Sigma \text{ Chord Set Products}. \qquad (39)$$

There are two cases for which i is certainly zero; $i=0$ for fundamental circuits and $i=0$ for the set of meshes ("windows") of a planar network. A detailed discussion of this question has been given by Cederbaum.[6] Since the network functions are independent of the circuit basis chosen, the fundamental system of circuits is chosen and $i=0$. In this report therefore $T\,2$ will be used in the form

$$T\,2' : \Delta_m = \Sigma \text{ Chord Set Products}, \qquad (40)$$

for networks without mutual inductances and for fundamental systems of circuits.

The topological formula $T\,2'$ was originally given by Kirchhoff[15] in the form:

Δ_m is the sum of products of resistances taken $e-v+1$ at a time, which have the common property that, when these elements are removed, no circuits remain.

The topological formulas $T\,1$ and $T\,2'$ can be combined by observing that

$$z_i \cdot y_i = 1 \text{ and so}$$

$$(z_1 \cdot z_2 \ldots z_e) \text{ (Tree Product)} = \text{Chord Set Product.}$$

$T\,3$: For a network without mutual inductances

$$\Delta_m = z_1 \cdot z_2 \cdot z_3 \ldots z_e \Delta_n, \tag{41}$$

where fundamental circuits are used.

$T\,3$ was originally given by Tsang [26] and since has been extended by Cederbaum[6] to networks containing magnetic coupling.

It is convenient to introduce a notation for sums of tree products and sums of chord set products. The node determinant expressed in terms of the element admittances $y_1, y_2, \ldots, y_e$ is a homogeneous polynomial of degree $v-1$ in these variables and is linear in any one y_j. Such a polynomial expression for a node determinant is known as the node discriminant[9] and is symbolized as $V(y_1, y_2, \ldots, y_e)$. The abbreviation $V(Y)$ will be used to denote $V(y_1, y_2, \ldots, y_e)$. The notation for the mesh discriminant is simplified by introducing the following complement convention of Percival.[20]

Given the polynomial $V(Y)$, the complementary polynomial $C[V(Y)]$ is formed by replacing each product in $V(Y)$ by the product of the variables not in this product. The polynomial $C[V(Z)]$ is obtained by replacing y_i by z_i in $C[V(Y)]$. With these conventions,

$$\Delta_n = V(Y) \tag{42}$$

$$\Delta_m = C[V(Z)]. \tag{43}$$

When this complement convention is used the complement of zero is zero.

8. Symmetrical Cofactors of the Node Determinants — 2-Trees

In this section the cofactor of an element on the main diagonal of the matrix $Y_n(s)$ is investigated. The cofactor of an element in the (i, i) position is obtained by deleting the i^{th} row and i^{th} column of the matrix $Y_n(s)$ and taking the determinant of the resultant matrix. Since

$$Y_n(s) = A\,Y_e\,A', \tag{44}$$

deleting the i^{th} row from $Y_n(s)$ is equivalent to deleting the i^{th} row from A. The matrix obtained by deleting the i^{th} row from A is denoted by A_{-i}. Similarly, deleting the i^{th} column from $Y_n(s)$ is equivalent to deleting the i^{th} column from A', that is, deleting the i^{th} row from A. Thus the cofactor of the (i, i) element is

$$\Delta_{ii} = \det\,(A_{-i} \cdot Y_e \cdot A'_{-i}). \tag{45}$$

The same technique that was applied to Δ_n can be applied to Δ_{ii}. However, it is better to construct the graph for which Δ_{ii} is the node determinant and apply rule $T\,1$. If the i^{th} vertex of the network N is shorted to the reference vertex and if this new combined vertex is used as the reference vertex, the node admittance matrix of the new network N_1 is precisely

$$Y_{n1} = A_{-i} \cdot Y_e \cdot A'_{-i}. \tag{46}$$

Thus Δ_{ii} is simply the sum of tree products for the graph obtained by identifying the i^{th} vertex with the reference vertex.

It is also instructive to examine the subgraphs of the network N which become the trees of the network N_1, so the formula can be extended to unsymmetrical minors. N_1 contains $(v-1)$ vertices and a tree of N_1 contains $v-2$ elements. The subgraph of N corresponding to such a tree of N_1 will not contain any circuits. Since it contains only $v-2$ elements it will not be connected, but it will be in two connected parts. One of the two parts may consist of an isolated vertex. The vertex i and the reference vertex will be in two different connected parts of this subgraph, in N. (If they were in the same connected part, shorting the i^{th} vertex with the reference vertex would produce a circuit.) Such a geometrical configuration has been named a 2-tree by Percival.[20] The formal definition of a 2-tree is as follows: A 2-tree is a pair of unconnected, circuitless subgraphs, each subgraph being connected, which together include all the vertices of the graph. One (or in trivial graphs, both) of the subgraphs may consist of an isolated vertex. The symbol T_2 is used for a 2-tree. Very often 2-trees in which certain designated vertices are required to be in different connected parts will be used. Then subscripts are used to denote such 2-trees.

For example,

$$T_{2_{ab,\,cde}}$$

is the symbol for a 2-tree in which the vertices a and b are in one connected part and the vertices c, d, and e are in the other part. A 2-tree product is the product of the admittances of the branches of a 2-tree. Again one of the two parts may be an isolated vertex. The product for an isolated vertex is defined to be 1. A 2-tree, such as

$$T_{2_{i,\,i}}$$

in which the same vertex i is required to be in different connected parts, has by definition a zero product.

A sum of 2-tree products, such as occurs in the expansion of a symmetrical cofactor of the node admittance matrix, is symbolized by $W(Y)$ with subscripts denoting any special vertices which are required to be in different parts. In terms of 2-trees, the formula for the symmetrical cofactor can be expressed as:

$T\,4$: If r is the reference vertex of node equations, the cofactor of an element in the (i, i) position is given by

$$\Delta_{ii} = \Sigma\ T_{2_{i,\,r}} \text{ products} \tag{47}$$

that is

$$\Delta_{ii} = W_{i,\,r}(Y) \tag{48}$$

9. Driving Point Admittance

It is instructive to review the procedure for computing the driving point impedance of a one terminal-pair passive network, such as Fig. 8, before stating the topological formula.

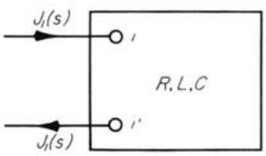


Fig. 8.

$J_1(s)$ signifies a current driver in the s-domain. The network contains no other drivers. The driving point admittance at vertices 1 and 1′ is defined as

$$Y_d(s) = J_1(s)/V_{11'}(s) \tag{49}$$

where $V_{11'}(s)$ is the transform of the voltage between vertices 1 and 1′ with a reference positive polarity at 1. All the initial currents and voltages are zero.

To compute $Y_d(s)$ it is convenient to write node equations with vertex 1′ as the reference vertex. These equations are

$$Y_n(s) \cdot V_n(s) = J_n(s) \tag{50}$$

or, in detail,

$$\begin{bmatrix} Y_{11} & Y_{13} & \ldots & Y_{1,\,v-1} \\ Y_{31} & Y_{33} & \ldots & Y_{3,\,v-1} \\ \cdot & \cdot & \cdot & \cdot \\ Y_{v-1,\,1} & Y_{v-1,\,3} & \ldots & Y_{v-1,\,v-1} \end{bmatrix} \begin{bmatrix} V_{11'} \\ V_{31'} \\ \cdot \\ V_{v-1,\,1'} \end{bmatrix} = \begin{bmatrix} J_1 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \tag{51}$$

From which the solution for $V_{11'}$ is

$$V_{11'} = (\Delta_{11}/\Delta)\,J_1 \tag{52}$$

or

$$Y_d(s) = \Delta/\Delta_{11}. \tag{53}$$

It is important to notice that the network has not been modified in the vertex matrix (and hence Y_n). Such a modification is done in mesh equations. Therefore, the topological formulas $T\,1$ and $T\,4$ can be used to get the topological formula for the driving point admittance as:

$T\,5$: For a network which contains no magnetic coupling

$$Y_d(s) = V(Y)/W_{1,\,1'}(Y) \tag{54}$$

where 1 and 1′ are the input vertices.

The computation of $V(Y)$ and $W_{1,\,1'}(Y)$ can be done without any regard to which vertex is used in writing the node equations. This is as it should be, since the driving point admittance is independent of the reference vertex chosen.

Example:

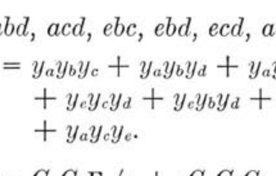


Fig. 9.

The trees of this network are:

abc, abd, acd, ebc, ebd, ecd, ade, ace.

Hence
$$V(Y) = y_ay_by_c + y_ay_by_d + y_ay_cy_d + y_ey_by_c + y_ey_cy_d + y_ey_by_d + y_ay_dy_e + y_ay_cy_e. \tag{55}$$

and so
$$\Delta_n = G_aG_b\Gamma_c/s + sG_aG_bC_d + G_aC_d\Gamma_c + G_b\Gamma_c\Gamma_e/s^2 + G_bC_d\Gamma_e + C_d\Gamma_c\Gamma_e/s + G_aC_d\Gamma_e + G_a\Gamma_c\Gamma_e/s^2 \tag{56}$$

that is,
$$\Delta_n = G_aG_bC_ds + (G_aC_d\Gamma_c + G_bC_d\Gamma_e + G_aC_d\Gamma_e) + (G_aG_b\Gamma_c + \Gamma_c\Gamma_eC_d)/s + (G_b\Gamma_c\Gamma_e + G_a\Gamma_c\Gamma_e)/s^2 \tag{57}$$

The 2-trees $T_{2_{1,\,1'}}$ are:

ac, ed, ae, ab, be, bc, cd, bd.

Hence

$$W_{1,1'}(Y) = y_a y_c + y_e y_d + y_a y_e + y_a y_b + y_b y_e + y_b y_c + y_c y_d + y_b y_d. \quad (58)$$

And so

$$\Delta_{11} = G_b C_d s + (\Gamma_e C_d + G_a G_b + \Gamma_c C_d) + (\Gamma_c G_a + G_a \Gamma_e + G_b \Gamma_e + G_b \Gamma_c)/s. \quad (59)$$

Hence

$$Y_d(s) = \{G_a G_b C_d s + (G_a C_d \Gamma_c + G_b C_d \Gamma_e + G_a C_d \Gamma_e) + (G_a G_b \Gamma_e + C_d \Gamma_c \Gamma_e)/s + (G_b \Gamma_c \Gamma_e + G_a \Gamma_c \Gamma_e)/s^2\}/\{G_b C_d s + (C_d \Gamma_e + G_a G_b + C_d \Gamma_c) + (G_a \Gamma_c + G_a \Gamma_e + G_b \Gamma_e + G_b \Gamma_c)/s\} \quad (60)$$

10. Symmetrical Cofactors of the Mesh Determinant

The cofactor of the element in the (i, i) position of the matrix $Z_m(s)$ is of interest only where there is at least one element in the i^{th} circuit which is not in any other circuit. Therefore it is assumed that there is an element y_j in circuit i which is in no other circuit. Let the vertices of y_j be 1 and 1′.

Using the same notation as before,

$$\Delta_{ii} = \det B_{-i} \cdot Z_e \cdot B'_{-i}. \quad (61)$$

Under the assumption that y_j is in no other circuit, the matrix $B_{-i} \cdot Z_e \cdot B'_{-i}$ will be the mesh impedance of the network obtained by deleting element y_j. If the network obtained by deleting element y_j is denoted as N_1, then

$$T\,6 : \Delta_{ii} = \Sigma \text{ Chord Set Products of } N_1 = C\,[V_1(Z)]. \quad (62)$$

11. Driving Point Impedance

It is well to review the procedure for computing the driving point impedance of a passive network. Given the network N, a voltage generator E_1 is connected at the input terminals, as in Fig. 10.

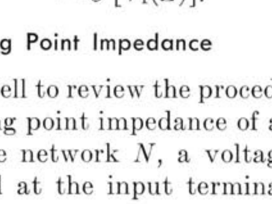


Fig. 10.

Considering E_1 and I_1 as Laplace transforms of the generator voltage and current, the driving point impedance is defined as:

$$Z_d(s) = E_1(s)/I_1(s); \text{ all initial values zero.} \quad (63)$$

In order to compute Z_d mesh equations are written for the system, selecting only one circuit, for example circuit 1, through the generator, as shown in Fig. 10. Then the mesh equations are:

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1\mu} \\ Z_{21} & Z_{22} & \cdots & Z_{2\mu} \\ \cdot & \cdot & \cdots & \cdot \\ Z_{\mu 1} & Z_{\mu 2} & \cdots & Z_{\mu\mu} \end{bmatrix} \begin{bmatrix} I_1 \\ I_{m2} \\ \cdot\cdot \\ I_{m\mu} \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \\ \cdot\cdot \\ 0 \end{bmatrix} \quad (64)$$

From which:

$$I_1 = (\Delta_{11}/\Delta)\,E_1 \quad (65)$$

so that

$$Z_d = \Delta/\Delta_{11} \quad (66)$$

As contrasted with the node system, the matrix Z_m is written not just for the network N, but for the network and the generator. The matrix Z_m for the network alone has one row and one column less than the matrix above. In fact, if the first row and column were deleted from the matrix of Eq. 64, the mesh impedance matrix of the network is obtained. Thus, writing the polynomial V for the network alone,

$$\Delta_{11} = C\,[V(Z)]; \quad (67)$$

The determinant Δ on the other hand is for N and the generator. However, the generator impedance is zero. Thus, for the coefficient matrix Z_m, the matrix for the network and the generator is the same as the matrix obtained for N, when the vertices 1 and 1′ are shorted. Therefore, applying formulas $T\,4$ and $T\,2$ jointly, and speaking with reference to the network N,

$$\Delta = C\,[W_{1,1'}(Z)]. \quad (68)$$

Thus, completing the topological formula for the driving point impedance:

T 7: For a network without mutual inductances, the driving point impedance at vertices 1 and 1′ is given by

$$Z_d(s) = C\,[W_{1,1'}(Z)]/C\,[V(Z)] \quad (69)$$

Example:

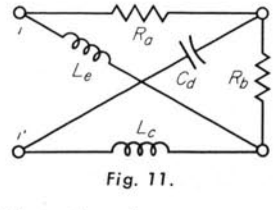


Fig. 11.

The trees of this network are:

abc, abd, acd, ace, ade, bce, cde, bde.

$$V(Z) = Z_a Z_b Z_c + Z_a Z_b Z_d + Z_a Z_c Z_d + Z_a Z_c Z_e + Z_a Z_d Z_e + Z_b Z_c Z_e + Z_c Z_d Z_e + Z_b Z_d Z_e. \quad (70)$$

Therefore

$$C[V(Z)] = Z_dZ_e + Z_cZ_e + Z_bZ_e + Z_bZ_d + Z_bZ_c + Z_aZ_d + Z_aZ_b + Z_aZ_c. \quad (71)$$

$$\Delta_{11} = CV(Z) = L_e/C_d + s^2L_cL_e + sL_eR_b + R_b/sC_d + sL_cR_b + R_a/sC_d + R_aR_b + sR_aL_c. \quad (72)$$

$$\Delta_{11} = s^2L_cL_e + s(L_eR_b + L_cR_b + R_aL_c) + R_aR_b + L_e/C_d + (R_b/C_d + R_a/C_d)/s. \quad (73)$$

The 2-trees $T_{2_{1,1'}}$ are

$$ab,\ ac,\ ae,\ bc,\ bd,\ be,\ cd,\ de.$$

$$W_{1,1'}(Z) = Z_aZ_b + Z_aZ_c + Z_aZ_e + Z_bZ_c + Z_bZ_d + Z_bZ_e + Z_cZ_d + Z_dZ_e. \quad (74)$$

$$C[W_{1,1'}(Z)] = Z_cZ_dZ_e + Z_bZ_dZ_e + Z_bZ_cZ_d + Z_aZ_dZ_e + Z_aZ_cZ_e + Z_aZ_cZ_d + Z_aZ_bZ_e + Z_aZ_bZ_c. \quad (75)$$

Therefore

$$\Delta = sL_cL_e/C_d + R_bL_e/C_d + R_bL_c/C_d + R_aL_e/C_d + s^2R_aL_cL_e + R_aL_c/C_d + sR_aR_bL_e + sR_aR_bL_c. \quad (76)$$

$$\Delta = s^2(R_aL_cL_e) + s(L_cL_e/C_d + R_aR_bL_e + R_aR_bL_c) + (R_bL_e/C_d + R_bL_c/C_d + R_aL_e/C_d + R_aL_c/C_d). \quad (77)$$

and

$$Z_d(s) = \{s^3(R_aL_cL_e) + s^2(L_cL_e/C_d + R_aR_bL_c + R_aR_bL_e) + (R_a + R_b)(L_e + L_c)/sC_d\}/\{s^3L_cL_e + s^2(L_eR_b + L_cR_b + R_aL_c) + s(R_aR_b + L_e/C_d) + (R_b/C_d + R_a/C_d)\} \quad (78)$$

12. Percival's Rules for Computation of Trees

Two basic rules can be stated for the computation of trees of a complicated network.

Rule 1: If $V_1(Y)$, $V_2(Y)$, . . . , $V_k(Y)$ are the tree admittance polynomials for the components G_1, G_2, . . . , G_k of a separable graph G, then the polynomial $V(Y)$ of G is given by

$$V(Y) = V_1(Y) \cdot V_2(Y) \cdot V_3(Y) \ldots V_k(Y). \quad (79)$$

Rule 2: If two subgraphs G_1, G_2 of a connected graph G have exactly two vertices i and j in common, then for G consisting of G_1 and G_2,

$$V(Y) = V_1(Y) \cdot W_{2_{i,j}}(Y) + V_2(Y) \cdot W_{1_{i,j}}(Y). \quad (80)$$

Rule 2 is seen to be valid by observing that every tree must contain a path between vertices i and j, either in G_1 or in G_2, but not in both. Thus every tree consists of a tree in G_1 and a 2-tree in G_2 or vice versa. Conversely, a tree of one of the subgraphs, and a 2-tree in the other separating vertices i, j, consitute a tree of G.

Rule 2 is useful in computation. An element of G can be chosen first as G_1. If this element is y_k, with vertices i and j,

$$V(Y) = y_k \cdot W_{i,j}(Y) + V_2(Y)$$

where W is now simply the 2-tree sum of the graph and V_2 is the sum of tree products when y_k is removed from the graph. Another element may be chosen to compute V_2 by the same method and the process repeated until the polynomial can be written down by inspection.

Example:

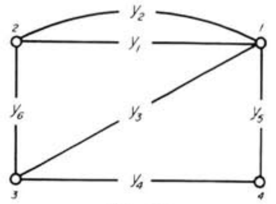


Fig. 12.

$$V(Y) = y_1W_{1,2}(Y) + V_2(Y) = y_1[y_3y_4 + y_3y_5 + y_4y_5 + y_4y_6 + y_5y_6] + V_2(Y) \quad (81)$$

$$V_2(Y) = y_2[y_3y_4 + y_3y_5 + y_4y_5 + y_4y_6 + y_5y_6] + V_3(Y) \quad (82)$$

$$V_3(Y) = y_3[y_4y_6 + y_5y_6] + y_4y_5y_6. \quad (83)$$

Hence

$$V(Y) = (y_1 + y_2)(y_3y_4 + y_3y_5 + y_4y_5 + y_4y_6 + y_5y_6) + y_3(y_4y_6 + y_5y_6) + y_4y_5y_6. \quad (84)$$

13. 2-Tree Identities

In order to state topological formulas in the simplest possible forms (without any redundant elements) a few 2-tree identities are used. Most of these are self-evident. First, observe that:

$$W_{i,j} = W_{j,i} \quad (85)$$

where every 2-tree with vertices i and j in different parts appears in both polynomials. The second useful identity is:

$$W_{i,j} = W_{i,jk} + W_{ik,j} \quad (86)$$

where k is any other vertex. This identity is seen to be true since k must be in one of the two connected parts. Equation 86 may also be stated in the more convenient form:

$$W_{i,j} - W_{ik,j} = W_{i,jk} \quad (87)$$

IV. TOPOLOGICAL FORMULAS FOR ADMITTANCES IN TWO TERMINAL-PAIR NETWORKS

14. Maxwell's Rule for Transfer Admittance

Maxwell[18] developed the original rule for the transfer function of a two terminal-pair network. Maxwell's rule for the current in an element between vertices r and s and oriented away from r, due to a voltage driver E with vertices p and q and with reference + at q, is

$$i_{rs} = Y_{rs} \cdot Y_{pq} \, (\Delta_{rs,\, pq}/\Delta) \cdot E. \tag{88}$$

In this formula the term $\Delta_{rs,\, pq}$ is the difference of cofactors selected from the node admittance matrix. Maxwell's rule for this factor is:

$\Delta_{rs,\, pq}$ is the sum of products of admittances, taken $v-2$ at a time, omitting all the terms which contain Y_{rs} or Y_{pq} and other terms either making closed circuits with themselves or with the help of Y_{rs} and Y_{pq}. The terms which contain Y_{qr} (or which form a closed circuit with Y_{qr}) and Y_{ps} (or those forming closed circuits with Y_{ps}) are taken as positive terms and similar terms with Y_{pr} and Y_{qs} are taken as negative terms.

Because each term contains $v-2$ factors and does not include a circuit, each product in Maxwell's formula corresponds to a 2-tree product. However, neither pair of vertices (p, q), (r, s) can be in the same connected part, since the terms which form closed circuits with Y_{pq}, Y_{rs} and those containing Y_{pq}, Y_{rs} are to be omitted. Thus, the 2-trees selected are simultaneously

$$T_{2_{p,\,q}} \text{ and } T_{2_{r,\,s}}$$

Therefore there are two possible sets of 2-trees to be selected:

$$T_{2_{pr,\,qs}} \text{ and } T_{2_{ps,\,qr}}.$$

Maxwell affixes a positive sign to the second set of 2-trees and a negative sign to the first set. Maxwell's rule is next stated in terms of 2-trees, after introducing the more common notation in the theory of two terminal-pair networks. Let Fig. 13 represent a two terminal-pair network with input vertices $(1, 1')$ and output vertices $(2, 2')$ with the references for the input and output current and voltage as shown. The load connected to the output terminals in the figure is y_L.

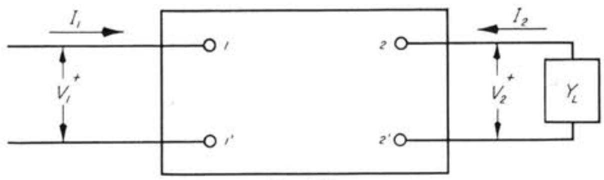


Fig. 13.

Let the node equations be written for this network with the vertex $1'$ as the reference vertex. These equations have the form:

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1,\, v-1} \\ y_{21} & y_{22} & \cdots & y_{2,\, v-1} \\ \cdot & \cdot & \cdots & \cdot \\ y_{v-1,\, 1} & y_{v-1,\, 2} & \cdots & y_{v-1,\, v-1} \end{bmatrix} \begin{bmatrix} V_{11'} \\ V_{21'} \\ \cdots \\ V_{v-1,\, 1'} \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ \cdot\cdot \\ 0 \end{bmatrix} \tag{89}$$

Then the output voltage $V_{22'} = V_2$ is given by

$$V_2 = \frac{\Delta_{12} - \Delta_{12'}}{\Delta} I_1. \tag{90}$$

Thus Maxwell's rule states that:

$$\Delta_{12} - \Delta_{12'} = \Sigma \, T_{2_{12,\, 1'2'}} \text{ Products} \\ - \Sigma \, T_{2_{12',\, 1'2}} \text{ Products} \tag{91}$$

15. Topological Formula for Unsymmetrical Cofactors

T 8: Let $1'$ be the reference vertex of a system of node equations, for a network which contains no magnetic coupling. Then the cofactor of an element in the (i, j) position is given by

$$\Delta_{ij} = W_{ij,\, 1'}(Y) = \Sigma \, T_{2_{ij,\, 1'}} \text{ Products}, \tag{92}$$

where the summation on the right is over all the 2-trees with vertices i and j in one connected part and vertex $1'$ in the other.

Proof of 8: The cofactor of an element in the (i, j) position is given by

$$\Delta_{ij} = (-1)^{i+j} M_{ij} \tag{93}$$

where M_{ij} is the determinant of a matrix obtained by deleting the i^{th} row and j^{th} column from the node admittance matrix Y_n. Hence

$$M_{ij} = \det (A_{-i} \cdot Y_e \cdot A_{-j}) \qquad (94)$$

where, as before, the subscript indicates the row which has been deleted from the incidence matrix. Just as in the case of the symmetrical minors, the non-zero majors of the matrix A_{-i} correspond one-one to the 2-trees of the network which have the vertex i in one connected part and the vertex $1'$ in the other. Similarly the non-zero majors of the matrix A_{-j} are in one-one correspondence with the 2-trees of the network which have the vertex j in one connected part and the vertex $1'$ in the other.

Using the Binet-Cauchy theorem[2]

$$M_{ij} = \Sigma \text{ Products of corresponding majors of } (A_{-i} \cdot Y_e) \text{ and } A'_{-j}. \qquad (95)$$

As before, the matrix product $A_{-i} \cdot Y_e$ differs from A_{-i} only in that the k^{th} column is multiplied by y_k, $k = 1, 2, \ldots, e$. Thus a non-zero major of $A_{-i}Y_e$ is (except possibly for sign) a 2-tree product of a 2-tree $T_{2_{i,1'}}$. If the corresponding major of A'_{-j} is also non-zero, to give a non-zero product in Eq. 95, this set of elements must also be a 2-tree $T_{2_{j,1'}}$. Thus the products in Eq. 95, which are non-zero, correspond to 2-trees in which both the vertices i and j are in one connected part and the reference vertex $1'$ is in the other; for example, subgraphs which are 2-trees

$$T_{2_{ij,1'}}.$$

In order to establish the sign to be prefixed to the 2-tree products, the submatrix consisting of the columns corresponding to the elements of one of the 2-trees of the type $T_{2_{ij,1'}}$ is selected from the incidence matrix A. This submatrix is of order $(v-1, v-2)$. Deleting row i from this matrix and calculating the determinant provides the major of A_{-i}, which is the sign of the major of $A_{-i} \cdot Y_e$. If row j is deleted from this matrix and the determinant is taken, the sign of the major of A_{-j} and hence of the corresponding major of A'_{-j} is similarly provided.

Each such 2-tree necessarily contains a path between the vertices i and j. Let this path from i to j consist of the elements

$$e_{q_1}, e_{q_2}, e_{q_3}, \ldots, e_{q_k}$$

in that order. In the chosen submatrix of A, the columns corresponding to these elements will have the following structure. Column q_1 will have a non-zero entry in row i. Columns q_1 and q_2 will have non-zero entries in the same row which is different from row i. Columns q_2 and q_3 will have non-zero entries in another common row, etc. Finally column q_k has a non-zero entry in row j. Two columns which have non-zero entries in the same row can be called adjacent, since they correspond to adjacent elements of the graph. Then, in the sequence of columns

$$q_1, q_2, \ldots, q_k,$$

only successive columns are adjacent. Using these results the submatrix of A is reduced to one in which column q_1 has non-zero entries in rows i and j and zeros in the other rows. This reduction is achieved by means of column operations only, so that the majors of A_{-i} and A_{-j} are left invariant under these operations.

Let column q_1 have a 1 in the i^{th} row; the case where this entry is -1 is the same and will not be considered. Column q_1 has a -1 in another row, for example r, and column q_2 has a non-zero entry in this row. If this entry is $+1$, add column q_2 to column q_1, but if this entry is -1, subtract column q_2 from column q_1. In either case, column q_1 has a $+1$ in row i, a -1 in another row (the row in which column q_2 has a non-zero entry) and zeros in all other rows. Using columns q_1 and q_3, if the common row entries have the same sign, subtract column q_3 from column q_1, but if they have opposite signs, add. After this the -1 in column q_1 is moved to a row in which column q_4 has a non-zero entry. After repeated application of this procedure the -1 finally moves to a row in which column q_k has a non-zero entry not adjacent to column q_{k-1}, namely row j. This gives a matrix in which column q_1 has a 1 in row i, a -1 in row j and zeros in all other rows. Let this final matrix be denoted as A_d.

There are two cases to consider: $i > j$ and $i < j$ and since the two cases are identical let $i > j$.

Consider the major of A_{-i} which is obtained by deleting row i from the matrix A_d obtained above and taking the determinant. This major is expanded by column q_1 which has only one non-zero entry, -1 in the j^{th} row. (Since $i > j$, the deleted row is below row j, so the row index of row j is unaltered.) Let the determinant of the matrix obtained by deleting rows i and j and column q_1 from A_d be denoted as D. Then

$$\begin{aligned}\text{Major of } A_{-i} &= (-1)^{q_1+j} \cdot (-1) \cdot D \\ &= (-1)^{q_1+j+1} \cdot D. \qquad (96)\end{aligned}$$

The major of A_{-j} is obtained by deleting row j from the matrix A_d and taking the determinant. Column q_1 of this determinant has a 1 in row $(i-1)$ and zeros in all other rows. (The row index of this row has decreased by one since row j has been deleted.) Expand the determinant by column q_1. The minor obtained by deleting column q_1 and row $i-1$ is the same determinant D that was obtained earlier. Hence,

$$\text{Major of } A_{-j} = (-1)^{q_1+i-1}(1) \cdot D. \qquad (97)$$

Therefore the product of the two majors of $(A_{-i} \cdot Y_e)$ and A'_{-j} is given by

$$(-1)^{2q_1+i+j} \cdot D^2 \cdot (T_{2_{ij},1'} \text{ Product}).$$

This is the same as

$$(-1)^{i+j} \cdot (T_{2_{ij},1'} \text{ Product})$$

since D is either 1 or -1, as it is selected from the incidence matrix A. The i and j are independent of the major selected from A_{-i} and A_{-j}, so

$$\begin{aligned}\det (A_{-i} \cdot Y_e \cdot A'_{-j}) &= (-1)^{i+j} \Sigma\, T_{2_{ij},1'} \text{ Products.} \\ &= (-1)^{i+j} \cdot W_{ij,1'}(Y). \qquad (98)\end{aligned}$$

Finally,

$$\Delta_{ij} = (-1)^{i+j} \det (A_{-i} \cdot Y_e \cdot A'_{-j}) = W_{ij,1'}(Y) \quad (99)$$

and the rule is proved.

With the help of this formula Maxwell's formula for the transfer admittance function can be established. The formula for the unsymmetrical cofactor contains, as a special case, the formula for a symmetrical cofactor. For, letting $i=j$ in formula T 8,

$$\Delta_{ii} = W_{ii,1'}(Y) = W_{i,1'}(Y) \qquad (100)$$

since the vertex i is always in the same part as itself.

16. Maxwell's Formula

T 9: If Y_n is the node admittance matrix of a network which does not contain any mutual inductances,

$$\Delta_{12} - \Delta_{12'} = W_{12,1'2'}(Y) - W_{12',1'2}(Y). \quad (101)$$

Formula T 9, which is Maxwell's formula for the denominator factor of the transfer admittance function is proved by observing that

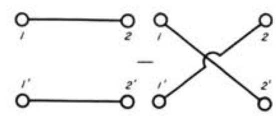
$$W_{12,1'}(Y) = W_{12,1'2'} + W_{122',1'} \qquad (102)$$

and

$$W_{12',1'}(Y) = W_{12',1'2} + W_{122',1'} \qquad (103)$$

The 2-trees of the form $T_{2_{122'},1'}$ cancel on subtraction.

Percival[20] expresses rule T 9 in the following intuitive fashion.

$W_{12',1'2'} - W_{12',1'2} =$ 

The argument above illustrates the typical character of all topological formulas — namely, no superfluous terms are calculated in following topological formulas, as in evaluating determinants. Only those terms which do not cancel are included.

As an example of the topological formula for the transfer function, a well-known result can be proven. Namely, that a ladder network without magnetic coupling is minimum phase[27] and the poles of the transfer admittance are the zeros of the series arms and the poles of shunt arms (admittances).

Figure 14 shows the general ladder network, where the series arms carry odd numbered subscripts and the shunt arms carry even numbered ones.

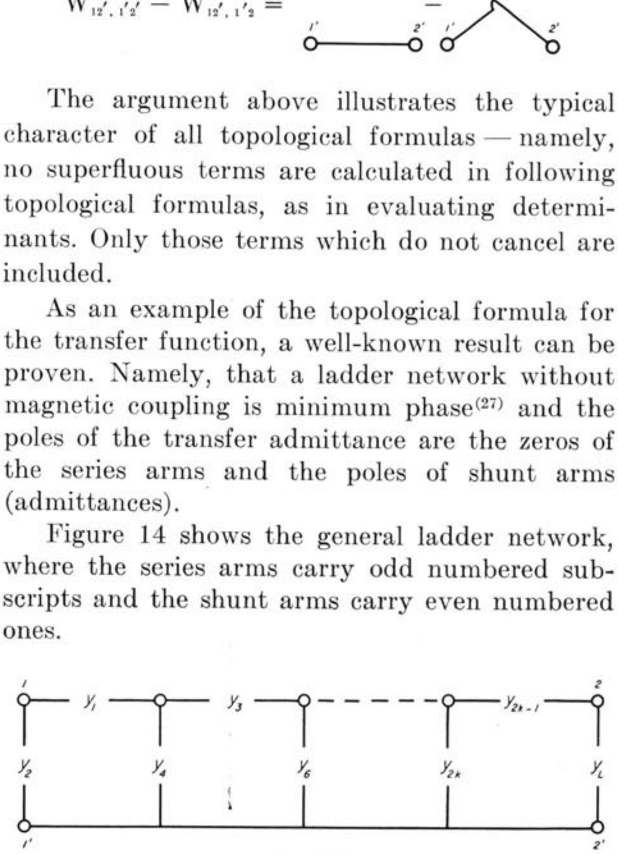


Fig. 14.

The transfer admittance then is given by the formula

$$\begin{aligned}V_{22'}/I_{11'} &= \Delta/\Delta_{12} \\ &= V(Y)/W_{12,1'} \qquad (104)\end{aligned}$$

since the second set of 2-trees in which the same vertex 1′, 2′ is required to be in two different connected parts is zero.

Now the vertex 2 has to be included in every tree of the network. Therefore, one of the admittances $y_2, y_4, \ldots, y_{2k}, y_L$ has to be included in each tree. Thus any pole of any one of these admittances is also a pole of sum $V(Y)$. This pole of course is also a pole of the transfer admittance (unless it is also a pole of $W_{12,\,1}$).

The other poles of the transfer admittance are the zeros $W_{12,\,1'}$. There is only one 2-tree product in this polynomial, namely

$$W_{12,\,1'}(Y) = y_1 \cdot y_3 \cdot y_5 \cdot \ldots \cdot y_{2k-1}. \tag{105}$$

The zeros of the denominator are simply the zeros of the series admittances and, since these are passive elements, their zeros are in the left half plane. Therefore, the function is a minimum phase function and the poles are complex frequencies for which the shunt arms are short-circuits (poles of admittance) and the series arms open circuits (zeros of admittance).

17. Short Circuit Admittance Functions

Two terminal-pair networks are more often described independently of the load y_L by means of the coefficient matrix of the system of equations

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{106}$$

The functions y_{ij} of this matrix are known as short circuit admittance functions since setting the appropriate voltage equal to zero makes the functions the ratio of current to voltage. The general node equations of the network of Fig. 15 may be written as

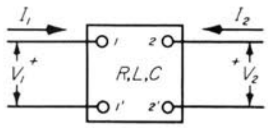


Fig. 15.

$$Y_n(s) \cdot V_n(s) = J_n(s) \tag{107}$$

where the vertex $1'$ is the reference vertex and the columns $V_n(s)$ and $J_n(s)$ are, respectively:

$$V_n = \begin{bmatrix} V_{11'} \\ V_{21'} \\ V_{2'1'} \\ \cdots \\ V_{v-1,\,1'} \end{bmatrix} \text{ and } J_n = \begin{bmatrix} I_1 \\ I_2 \\ -I_2 \\ 0 \\ \cdot\ \cdot \\ 0 \end{bmatrix} \tag{108}$$

On solving these equations for V_n we obtain:

$$\begin{bmatrix} V_{11'} \\ V_{21'} \\ V_{2'1'} \\ \cdots \\ V_{v-1,\,1'} \end{bmatrix} = \frac{1}{\Delta} \cdot \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{12'} & \ldots & \Delta_{1,\,v-1} \\ \Delta_{12} & \Delta_{22} & \Delta_{22'} & \ldots & \Delta_{2,\,v-1} \\ \Delta_{12'} & \Delta_{22'} & \Delta_{2'2'} & \ldots & \Delta_{2',\,v-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{1,\,v-1} & \Delta_{2,\,v-1} & \Delta_{2',\,v-1} & \ldots & \Delta_{v-1,\,v-1} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ -I_2 \\ 0 \\ \cdot\ \cdot \\ 0 \end{bmatrix} \tag{109}$$

All the entries of J_n are zero except the first three equations, which give

$$\begin{bmatrix} V_{11'} \\ V_{21'} \\ V_{2'1'} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{12'} \\ \Delta_{12} & \Delta_{22} & \Delta_{22'} \\ \Delta_{12'} & \Delta_{22'} & \Delta_{2'2'} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ -I_2 \end{bmatrix}. \tag{110}$$

Since $V_1 = V_{11'}$ and $V_2 = V_{22'} = V_{21'} - V_{2'1'}$ the functions V_1 and V_2 are

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\Delta} \cdot \begin{bmatrix} \Delta_{11} & \Delta_{12} - \Delta_{12'} \\ \Delta_{12} - \Delta_{12'} & \Delta_{22} + \Delta_{22'} - 2\Delta_{22'} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{111}$$

which may be written as $V_{2TP} = Z_{oc} \cdot I_{2TP}$. (112)

This last pair of equations can be solved for I_1 and I_2 to give the matrix of the short circuit admittance functions in terms of the determinants and cofactors of the node admittance matrix. The determinant of the coefficient matrix above is:

$$\begin{aligned} \det Z_{oc} &= \frac{1}{\Delta^2} \{\Delta_{11}\Delta_{22} + \Delta_{11}\Delta_{2'2'} - 2\Delta_{11}\Delta_{22'} \\ &\qquad - \Delta_{12}^2 - \Delta_{12'}^2 + 2\Delta_{12}\Delta_{12'}\} \\ &= \frac{1}{\Delta^2} \{(\Delta_{11}\Delta_{22} - \Delta_{12}^2) + (\Delta_{11}\Delta_{2'2'} - \Delta_{12'}^2) \\ &\qquad - 2(\Delta_{11}\Delta_{22'} - \Delta_{12'}\Delta_{21})\}. \end{aligned} \tag{113}$$

Using the determinantal identity[2]

$$\Delta_{ab}\Delta_{cd} - \Delta_{ad}\Delta_{cb} = \Delta\Delta_{ab,\,cd} \tag{114}$$

where a and c are any two rows and b and d are any two columns, $\Delta_{ab,\,cd}$ is the determinant obtained by deleting rows a, c and columns b, d. The expression above can be reduced to

$$\det Z_{oc} = \frac{1}{\Delta^2}\{\Delta\Delta_{1122} + \Delta\Delta_{112'2'} - 2\Delta\Delta_{1122'}\} \tag{115}$$

The solution for I_1, I_2 is finally:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'}} \cdot \begin{bmatrix} \Delta_{22} + \Delta_{2'2'} - 2\Delta_{22'} & \Delta_{12'} - \Delta_{12} \\ \Delta_{12'} - \Delta_{12} & \Delta_{11} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{116}$$

or

$$I_{2TP} = Y_{sc} \cdot V_{2TP} \tag{117}$$

where Y_{sc} is the matrix of the short circuit functions. All the cofactors above can be expressed in terms of 2-tree products except those in which two rows and columns have been deleted. In order to express these terms topologically a 3-tree product is defined.

A 3-tree is a set of $v-3$ elements which do not contain a circuit. Thus a 3-tree is a set of three unconnected circuitless subgraphs which together include all the vertices of the graph. One or two of these subgraphs may consist of isolated vertices. A 3-tree product is the product of the admittances of a 3-tree. As before the product for an isolated vertex will be 1. Certain specified vertices may also be required to be in different connected parts of the 3-tree. Such a 3-tree is denoted as

$$T_{3_{ab,\,c,\,def}}$$

in which the vertex sets (a, b), (c), (d, e, f) are required to be in different connected parts. The sum of 3-tree products is denoted by the symbol $U(Y)$ with subscripts on U to denote any specified distribution of vertices. Using arguments similar to those of $T\,8$, it is seen that

$$\Delta_{1122} = U_{1,\,2,\,1'} \tag{118}$$

$$\Delta_{112'2'} = U_{1,\,2',\,1'} \tag{119}$$

$$\Delta_{1122'} = U_{1,\,22',\,1'} \tag{120}$$

since 1′ is the reference vertex. The 3-trees of the form $T_{3_{1,\,22',\,1'}}$ will occur in both $U_{1,\,2',\,1'}$ and in $U_{1,\,2,\,1'}$. Such terms therefore will cancel in the det Z_{oc} expansion because of the $-2\Delta_{1122'}$ term. Therefore:

$$\begin{aligned} \Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'} = U_{12',\,2,\,1'} &+ U_{1,\,2,\,1'2'} \\ &+ U_{12,\,2',\,1'} \\ &+ U_{1,\,2',\,1'2} \end{aligned} \tag{121}$$

The other entries of Y_{sc} are:

$$\begin{aligned} \Delta_{22} + \Delta_{2'2'} - 2\Delta_{22'} &= W_{2,\,1'} + W_{2',\,1'} - 2W_{22',\,1'} \\ &= W_{2,\,1'2'} + W_{2',\,1'2} \\ &= W_{2,\,2'} \end{aligned} \tag{122}$$

$$\begin{aligned} \Delta_{12'} - \Delta_{12} &= W_{12',\,1'} - W_{12,\,1'} \\ &= W_{12',\,1'2} - W_{12,\,1'2'}. \end{aligned} \tag{123}$$

$$\Delta_{11} = W_{1,\,1'}. \tag{124}$$

In the sequel the abbreviation ΣU will be used for the sum

$$U_{12',\,2,\,1'} + U_{1,\,2,\,1'2'} + U_{12,\,2',\,1'} + U_{1,\,2',\,1'2}.$$

$T\,10$: For a two terminal-pair network which contains no mutual inductances, the matrix of the short circuit admittances is given by

$$Y_{sc} = \frac{1}{\Sigma U}\begin{bmatrix} W_{2,\,2'} & W_{12',\,1'2} - W_{12,\,1'2'} \\ W_{12',\,1'2} - W_{12,\,1'2'} & W_{1,\,1'} \end{bmatrix} \tag{125}$$

From the computation that was performed for det Z_{oc}, the topological formula can be written for the determinant of the short circuit admittance matrix since

$$Y_{sc} = Z_{oc}^{-1} \text{ and so } \det Y_{sc} = 1/(\det Z_{oc}).$$

$T\,11$: For a two terminal-pair network which contains no mutual inductances, the determinant of the short circuit admittance matrix is given by

$$\begin{aligned} \det Y_{sc} &= V(Y)/[U_{12',\,2,\,1'} + U_{1,\,2,\,1'2'} \\ &\quad + U_{12,\,2',\,1'} + U_{1,\,2',\,1'2}] \\ &= V(Y)/\Sigma\, U(Y). \end{aligned} \tag{126}$$

$T\,12$: In a two terminal-pair network which contains no mutual inductances, the open circuit impedance matrix is given by

$$Z_{oc} = \frac{1}{V(Y)} \cdot \begin{bmatrix} W_{1,\,1'}(Y) & W_{12,\,1'2'}(Y) - W_{12',\,1'2}(Y) \\ W_{12,\,1'2'}(Y) - W_{12',\,1'2}(Y) & W_{2,\,2'}(Y) \end{bmatrix}$$

$T\,13$: For a two terminal-pair network which contains no mutual inductances, the determinant of the open circuit impedance matrix Z_{oc} is given by

$$\det Z_{oc} = \Sigma\, U(Y)/V(Y).$$

Example 1

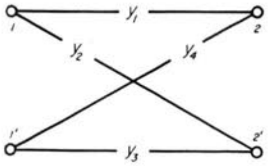


Fig. 16.

$$U_{12',\,2,\,1'} = y_2,\quad U_{1,\,2,\,1'2'} = y_3,\ U_{12,\,2',\,1'} = y_1,$$
$$U_{1,\,2',\,1'2} = y_4;$$
$$W_{2,\,2'} = (y_1+y_2)(y_3+y_4),\ W_{1,\,1'} = (y_1+y_4)(y_2+y_3);$$
$$W_{12',\,1'2} = y_2y_4,\quad W_{12,\,1'2'} = y_1y_3. \qquad (127)$$

Therefore the short circuit admittance matrix is:

$$Y_{sc} = \frac{1}{(y_1+y_2+y_3+y_4)} \cdot \begin{bmatrix} (y_1+y_2)\,(y_3+y_4) & y_2y_4-y_1y_3 \\ y_2y_4-y_1y_3 & (y_1+y_4)\,(y_2+y_3) \end{bmatrix} \qquad (128)$$

Example 2

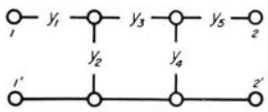


Fig. 17.

$$U_{12',\,2,\,1'} = U_{12,\,1',\,2'} = U_{1,\,2',\,1'2} = 0;$$
$$U_{1,\,2,\,1'2'} = (y_1+y_2)\,(y_3+y_4+y_5)+y_3(y_4+y_5);$$
$$W_{12',\,1'2} = 0,\quad W_{12,\,1'2'} = y_1y_3y_5;$$
$$W_{1,\,1'} = y_1y_3y_5+y_2y_3y_5+y_1y_4y_5+y_2y_4y_5+y_3y_4y_5;$$
$$W_{2,\,2'} = y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_1y_2y_5+y_1y_3y_5. \qquad (129)$$

Therefore

$$Y_{sc} = \frac{1}{(y_1+y_2)(y_3+y_4+y_5)+y_3(y_4+y_5)} \cdot \begin{bmatrix} y_1y_2(y_3+y_4+y_5) + y_1y_3(y_4+y_5) & -y_1y_3y_5 \\ -y_1y_3y_5 & y_3y_5(y_1+y_2+y_4) + y_4y_5(y_1+y_2) \end{bmatrix} \qquad (130)$$

V. TOPOLOGICAL FORMULAS FOR VOLTAGE AND CURRENT RATIOS AND EQUIVALENT T AND π NETWORKS

18. Voltage Ratio Transfer Function

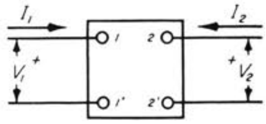


Fig. 18.

The two terminal-pair network of Fig. 18 ends in an open circuit so that $I_2=0$. Under these conditions the ratio

$$V_2/V_1$$

is defined as the voltage ratio transfer function from end 1 to end 2. Present practice in network theory is to use the symbol G for both current and voltage ratio transfer functions for both directions of transmission. This usage is unfortunate especially as the functions are not the same for both directions of transmission. Thus if G_{12} denotes the voltage ratio transfer function from end 1 to end 2 and G_{21} the voltage ratio from end 2 to end 1,

$$G_{12} \neq G_{21}. \tag{131}$$

For these reasons it seems desirable to borrow symbols from unilateral network theory for these functions. In this report the symbol μ is used for the voltage ratio transfer function and the symbol α for the current ratio transfer function. Subscripts are used to designate the input and output. Thus

$$\mu_{12} = V_2/V_1 \tag{132}$$

when end 2 is open circuited. And

$$\mu_{21} = V_1/V_2 \tag{133}$$

when end 1 is open. Similar conventions apply to current ratio transfer functions.

The following equations were derived in the last chapter for the two terminal-pair network.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta_{11} & \Delta_{12}-\Delta_{12'} \\ \Delta_{12}-\Delta_{12'} & \Delta_{22}+\Delta_{2'2'}-2\Delta_{22'} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$= \frac{1}{V(Y)}\begin{bmatrix} W_{1,1'} & W_{12,1'2'}-W_{12',1'2} \\ W_{12,1'2'}-W_{12',1'2} & W_{2,2'} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{134}$$

When end 2 is open circuited, $I_2=0$. Then,

$$V_1=(W_{1,1'}/V)\, I_1 \tag{135}$$

$$V_2=[(W_{12,1'2'}-W_{12',1'2})/V]\, I_1 \tag{136}$$

Therefore,

$$\mu_{12} = \frac{V_2}{V_1}\Bigg|_{I_2=0.} = (W_{12,1'2'}-W_{12',1'2})/W_{1,1'} \tag{137}$$

Similarly μ_{21} is obtained by setting $I_1=0$ and computing V_1/V_2. This produces the topological formula for the voltage ratio transfer functions.

T 14: The voltage ratio transfer functions of a two terminal-pair network without mutual inductances are given by:

$$\mu_{12} = \frac{W_{12,1'2'}(Y) - W_{12',1'2}(Y)}{W_{1,1'}(Y)} \tag{138}$$

$$\mu_{21} = \frac{W_{12,1'2'}(Y) - W_{12',1'2}(Y)}{W_{2,2'}(Y)} \tag{139}$$

Example:

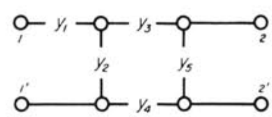


Fig. 19.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{V(Y)}\begin{bmatrix} W_{1,1'} & W_{12,1'2'}-W_{12',1'2} \\ W_{12,1'2'}-W_{12',1'2} & W_{2,2'} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{140}$$

$$\begin{aligned} V(Y) &= y_1y_2y_3y_4 + y_1y_2y_3y_5 + y_1y_2y_4y_5 + y_1y_3y_4y_5; \\ W_{1,1'} &= (y_1+y_2)(y_3y_4+y_4y_5+y_3y_5) + y_3y_4y_5; \\ W_{2,2'} &= y_1(y_2y_3+y_3y_4+y_2y_4); \\ W_{12,1'2'} &= y_1y_3y_4; \quad W_{12',1'2} = 0. \end{aligned} \tag{141}$$

Therefore,

$$Z_{oc} = \frac{1}{y_1(y_2y_3y_4+y_2y_3y_5+y_2y_4y_5+y_3y_4y_5)} \cdot \begin{bmatrix} (y_1+y_2)(y_3y_4+y_3y_5+y_4y_5)+y_3y_4y_5 & y_1y_3y_4 \\ y_1y_3y_4 & y_1(y_2y_3+y_3y_4+y_2y_4) \end{bmatrix} \tag{142}$$

The voltage ratio transfer functions are:

$$\begin{aligned} \mu_{12} &= (W_{12,1'2'} - W_{12',1'2})/W_{1,1'} \\ &= (y_1y_3y_4)/[(y_1+y_2)(y_3y_4+y_3y_5+y_4y_5) \\ &\quad + y_3y_4y_5] \end{aligned} \tag{143}$$

$$\mu_{21} = (W_{12,\,1'2'} - W_{12',\,1'2})/W_{2,\,2'}$$
$$= (y_1y_3y_4)/(y_1y_2y_3+y_1y_3y_4+y_1y_2y_4) \qquad (144)$$

19. Current Ratio Transfer Functions

The current ratio transfer function is defined as the ratio of the output to input currents when the output terminals are shorted. Thus

$$\alpha_{12} = \frac{I_2}{I_1} \Bigg|\ V_2 = 0 \qquad (145)$$

$$\alpha_{21} = \frac{I_1}{I_2} \Bigg|\ V_1 = 0. \qquad (146)$$

To compute the α's the network is considered as driven by a voltage generator at the non-shorted end. The equation relating voltages and currents at the terminals of the two terminal-pair network may be written as (from Chapter III)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$= \frac{1}{\Sigma U} \begin{bmatrix} W_{2,\,2'} & W_{12',\,1'2} - W_{12,\,1'2'} \\ W_{12',\,1'2} - W_{12,\,1'2'} & W_{1,\,1'} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (147)$$

where, as before,

$$\Sigma U = U_{12',\,2,\,1'} + U_{1,\,2,\,1'2'} + U_{12,\,2',\,1'} + U_{1,\,2',\,1'2} \qquad (148)$$

By the same procedure that was used for the voltage ratio transfer functions the topological formulas for the current ratios are derived.

T 15: For a network without mutual inductances, the current ratio transfer functions are given by

$$\alpha_{12} = \frac{I_2}{I_1} \Bigg|\ V_2 = 0 = \frac{W_{12',\,1'2}(Y) - W_{12,\,1'2'}(Y)}{W_{2,\,2'}(Y)} \qquad (149)$$

$$\alpha_{21} = \frac{I_1}{I_2} \Bigg|\ V_2 = 0 = \frac{W_{12',\,1'2}(Y) - W_{12,\,1'2'}(Y)}{W_{1,\,1'}(Y)} \qquad (150)$$

Comparing *T* 12 and *T* 13 the following relation is found between μ's and the α's.

$$\alpha_{12} = -\mu_{21} \qquad (151)$$

$$\alpha_{21} = -\mu_{12}. \qquad (152)$$

Example

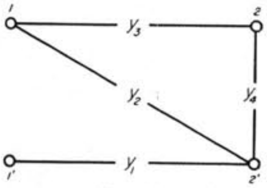


Fig. 20.

$$U_{12',\,2,\,1'} = y_2;\ U_{12,\,2',\,1'} = y_3;$$
$$U_{1,\,2,\,1'2'} = y_1;\ U_{1,\,2',\,1'2} = 0;$$
$$W_{2,\,2'} = y_1y_2 + y_1y_3;$$
$$W_{1,\,1'} = (y_1 + y_2)(y_3 + y_4) + y_3y_4;$$
$$W_{12,\,1'2'} = y_1y_3;\ W_{12',\,1'2} = 0. \qquad (153)$$

As a result

$$y_{sc} = \frac{1}{(y_1+y_2+y_3)} \cdot \begin{bmatrix} y_1(y_2+y_3) & -y_1y_3 \\ -y_1y_3 & (y_1+y_2)(y_3+y_4)+y_3y_4 \end{bmatrix} \qquad (154)$$

Therefore the current ratio transfer functions are

$$\alpha_{12} = (W_{12',\,1'2} - W_{12,\,1'2'})/W_{2,\,2'}$$
$$= (y_1y_3)/[y_1(y_2+y_3)] = -y_3/(y_2+y_3) \qquad (155)$$

$$\alpha_{21} = (W_{12',\,1'2} - W_{12,\,1'2'})/W_{1,\,1'}$$
$$= (-y_1y_3)/[(y_1+y_2)(y_3+y_4)+y_3y_4]. \qquad (156)$$

20. Equivalent T and π Networks

The *T* equivalent of a given two terminal-pair network is obtained by identifying the two terminal-pair equations

$$V_{2TP} = Z_{oc}\, I_{2TP}$$

that is,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad (157)$$

as the equations of the *T* network of Fig. 21, in which the elements may not be physically realizable.

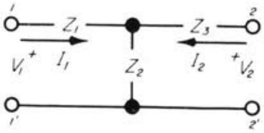


Fig. 21.

Comparing the terms

$$Z_1 = z_{11} - z_{12} \qquad (158)$$

$$Z_2 = z_{12} \qquad (159)$$

$$Z_3 = z_{22} - z_{12}. \qquad (160)$$

Thus the topological formulas for the elements of the *T*-equivalent are immediately derived. The formulas are written for the admittances of the *T*-network $Y_1 = 1/Z_1$, $Y_2 = 1/Z_2$ and $Y_3 = 1/Z_3$ for the sake of uniformity.

T 16: The admittances of the equivalent *T* of a given two terminal-pair network without mutual inductances, are given by

$$Y_1 = V(Y)/[W_{1,1'} - W_{12,1'2'} + W_{12,1'2}]$$
$$= V(Y)/[W_{122',1'} + W_{1,1'22'} + 2W_{12',1'2}] \quad (161)$$

$$Y_2 = V(Y)/[W_{12,1'2'} - W_{12',1'2}] \quad (162)$$

$$Y_3 = V(Y)/[W_{11'2,2'} + W_{2,11'2'} + 2W_{12',1'2}] \quad (163)$$

The equivalent π of a given two terminal-pair network is similarly obtained by examining the equations

$$I_{2TP} = Y_{sc} V_{2TP} \quad (164)$$

which are the node equations of the π network of Fig. 22. Once again the functions of the equivalent network may not be physically realizable.

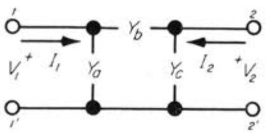


Fig. 22.

By comparing terms it is seen that

$$Y_a = y_{11} + y_{12} \quad (165)$$

$$Y_b = -y_{12} \quad (166)$$

$$Y_c = y_{22} + y_{12} \quad (167)$$

The topological formulas for the elements can be immediately established from the formulas for the short circuit admittance functions.

T 17: The admittances of the elements of the π network equivalent to a given two terminal-pair network without mutual coupling are given by

$$Y_a = [W_{2,2'} + W_{12',1'2} - W_{12,1'2'}]/\Sigma U$$
$$= [W_{11'2,2'} + W_{2,11'2'} + 2W_{12',1'2}]/\Sigma U \quad (168)$$

$$Y_b = [W_{12,1'2'} - W_{12',1'2}]/\Sigma U \quad (169)$$

$$Y_c = [W_{122',1'} + W_{1,1'22'} + 2W_{12',1'2}]/\Sigma U \quad (170)$$

where

$$\Sigma U = U_{12',2,1'} + U_{1,2,1'2'} + U_{12,2',1'}$$
$$+ U_{1,2',1'2} \quad (171)$$

Example: *T* Equivalent.

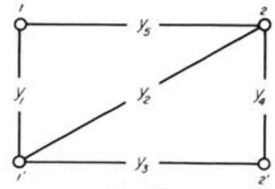


Fig. 23.

$$W_{122',1'} = y_4y_5;\ W_{1,1'22'} = y_2y_3+y_3y_4+y_2y_4;$$
$$W_{12',1'2} = 0;$$
$$W_{12,1'2'} = y_3y_5;\ W_{11'2,2'} = y_1y_5+y_1y_2+y_2y_5;$$
$$W_{2,11'2'} = y_1y_3;$$
$$V(Y) = y_1[(y_2+y_5)(y_3+y_4)+y_3y_4]$$
$$+ y_5(y_2y_3+y_3y_4+y_2y_4). \quad (172)$$

The elements of the *T* equivalent are given by

$$Y_1 = \{y_1[(y_2 + y_5)(y_3 + y_4) + y_3y_4]$$
$$+ y_5(y_2y_3 + y_3y_4 + y_2y_4)\}$$
$$/(y_4y_5 + y_2y_3 + y_2y_4 + y_3y_4) \quad (173)$$

$$Y_2 = \{y_1[(y_2 + y_5)(y_3 + y_4) + y_3y_4]$$
$$+ y_5(y_2y_3 + y_3y_4 + y_2y_4)\}/y_3y_5 \quad (174)$$

$$Y_3 = \{y_1[(y_2 + y_5)(y_3 + y_4) + y_3y_4]$$
$$+ y_5(y_2y_3 + y_3y_4 + y_2y_4)\}$$
$$/(y_1y_2 + y_1y_5 + y_2y_5 + y_1y_3) \quad (175)$$

Example: π Equivalent.

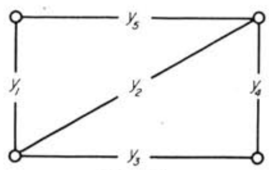


Fig. 24.

$$W_{11'2,2'} = y_1y_2 + y_1y_5 + y_2y_5;$$
$$W_{2,11'2} = y_1y_3;\ W_{122',1'} = y_4y_5;$$
$$W_{1,1'22'} = y_2y_3 + y_3y_4 + y_2y_4;$$
$$W_{12,1'2'} = y_3y_5;\ W_{12',1'2} = 0;$$
$$U_{12',2,1'} = 0;\ U_{1,2,1'2'} = y_3;$$
$$U_{12,1',2'} = y_5;\ U_{1,2',1'2} = y_2. \quad (176)$$

Thus the elements of the π equivalent are given by

$$Y_a = [y_1(y_2+y_3+y_5)+y_2y_5]/(y_2+y_3+y_5) \quad (177)$$

$$Y_b = (y_3y_5)/(y_2+y_3+y_5) \quad (178)$$

$$Y_c = [y_4(y_2+y_3+y_5)+y_2y_3]/(y_2+y_3+y_5) \quad (179)$$

VI. TOPOLOGICAL FORMULAS FOR TWO TERMINAL-PAIR NETWORKS IN TERMS OF IMPEDANCES OF NETWORK ELEMENTS

21. Kirchhoff's Rules

Kirchhoff[15] gave the following rules for the computation of the current I_{rs} in an element between vertices r and s with reference from r to s, due to a generator E between vertices p and q with a reference + at q:

$$I_{rs} = E_{qp}\,\Delta_{ab}/\Delta. \tag{180}$$

Δ is the mesh determinant which has already been considered in Chap. II.

Δ_{ab} is the sum of (signed) products of impedances taken $(e-v)$ at a time when they have the common property that after these elements have been removed there is only one circuit left. This circuit contains both the generator E and the element in which the current is being computed.* The terms by which the remaining circuit goes through both E_{qp} and I_{rs} in the same relative direction are given a positive sign, and those for which the remaining circuit goes through E and I_{rs} in opposite directions are given a negative sign. (The orientation refers to the element orientation.)

In order to correlate Kirchhoff's rules with 2-trees, it is convenient to introduce the conventions shown in Fig. 25.

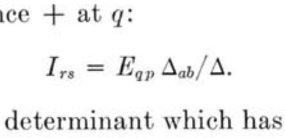


Fig. 25.

Let N denote the two terminal-pair network of Fig. 25, excluding the generator E and the load Z_L. N consists of R, L, C elements only. Consider one of the products in Δ_{ab} of Kirchhoff's rule which has $e-v$ elements, where e is the number of elements in the complete network, including E and Z_L. When this set of elements is removed, there are v elements remaining. This set of elements includes exactly one circuit which contains both E and Z_L. Therefore, if either E or Z_L (but not both) is removed, the rest is a tree of $(N+E+Z_L)$. If both E and Z_L are removed the rest is a 2-tree of N, which separates the vertices of E as well as the vertices of Z_L. Thus the remainder is both

$$T_{2_{1,1'}} \text{ and } T_{2_{2,2'}}.$$

Once again the 2-tree may be either

$$T_{2_{12,1'2'}} \text{ or } T_{2_{12',1'2}}.$$

The products in Δ_{ab} consist of the elements of N which are not in these 2-trees. Thus Δ_{ab} contains

$$C\,[T_{2_{12,1'2'}}] \text{ impedance products and}$$

$$C\,[T_{2_{12',1'2}}] \text{ impedance products,}$$

where C denotes complementation with respect to N only. Kirchhoff affixes a positive sign to the products of the first type and a negative sign to the products of the second type. So, by Kirchhoff's formula,

$$\Delta_{ab} = C\,[W_{12,\,1'2'}(Z)] - C\,[W_{12',\,1'2}(Z)]. \tag{181}$$

Complementation is with respect to N. The Z in parentheses implies that the products are impedance products.

22. Proof of Kirchhoff's Formula

Kirchhoff's formula can be proven by observing first:

A set of fundamental circuits can always be chosen for the complete network $(N+E+Z_L)$, such that E and Z_L are each only in one circuit. They may both be in the same fundamental circuit or in different ones.

If the network N is connected, a tree of N appears. The elements E and Z_L are chords for such a tree and so are in only one circuit each and in different circuits. If N is not connected, $(N+E)$ is connected. Otherwise $(N+E+Z_L)$ would be separable — it is assumed here that it is nonseparable. The element E which is in the tree of $(N+E)$ would be only in the fundamental circuit of Z_L. Thus E and Z_L are in only one circuit.

The latter case where E and Z_L are in the same circuit, is the driving point case that was given earlier in Chap. III. Therefore, it will be assumed that they are in different fundamental

* In this connection an error in Ku's paper[17] should be noted. Ku requires that the circuit contain either Z_{rs} or Z_{pq}. It should include both. The error is probably an error in translation from the original German text. The sign convention of Ku for these products is also ambiguous.

circuits. Let E be put in circuit 1 and Z_L in circuit 2 for notational convenience and oriented as shown in Fig. 26.

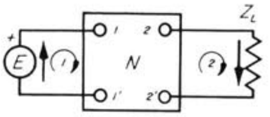


Fig. 26.

Then obviously,

$$I_L = (\Delta_{12}/\Delta)\,E, \tag{182}$$

with reference to the mesh equations. Since fundamental circuits were chosen,

$$\Delta = \Sigma \text{ Chord set products of } (N+E+Z_L) \tag{183}$$

(without any factor 2^{2i}) by the results of Chap. III.

$$\begin{aligned}\Delta_{12} &= \text{Cofactor of the } (1,2) \text{ element of } B_f Z_e B'_f \\ &= (-1)^{1+2} \det B_{-1} \cdot Z_e \cdot B'_{-2}\end{aligned} \tag{184}$$

using the same notation as in Chap. IV, the subscript denoting the deleted row.
Once again

$$\det B_{-1} \cdot Z_e \cdot B_{-2} = \Sigma \text{ products of corresponding majors of } (B_{-1} \cdot Z_e) \text{ and } B'_{-2}. \tag{185}$$

Deleting row 1 from B_f yields the circuit matrix of the network when circuit 1 is destroyed, which is obtained by deleting element E. Thus

Non-zero majors of B_{-1} are in one-one correspondence with chord sets of $(N+Z_L)$.

Similarly deleting row 2 of B_f yields the circuit matrix when circuit 2 is destroyed. This is the same as deleting element Z_L. Thus, non-zero majors of B_{-2} are in one-one correspondence with the chord sets of $(N+E)$.

Since Z_e is a diagonal matrix it is not considered.

To get a non-zero product of the two majors, the set of elements must be a chord set of both $(N+E)$ and $(N+Z_L)$. Thus the chord set cannot include either E or Z_L and E in $(N+E)$ and Z_L in $(N+Z_L)$ must be branches of the trees for which this is a chord set. The elements of N which are branches for these trees must, therefore, constitute a 2-tree of N. This 2-tree separates the vertices of both E and Z_L, so it is a 2-tree of one of the two types

$$T_{2_{12,\,1'2'}} \text{ or } T_{2_{12',\,1'2}}.$$

Conversely, the product of the elements in the complement in N of every such 2-tree is a term in det $(B_{-1} \cdot Z_e \cdot B'_{-2})$, since each such 2-tree with E is a tree of $(N+E)$ and with Z_L is a tree of $(N+Z_L)$. It remains to establish the signs of

$$C\,[W_{12,\,1'2'}(Z)] \text{ and } C\,[W_{12',\,1'2}(Z)].$$

A procedure is followed similar to the one adopted in establishing Maxwell's rule for unsymmetrical cofactors of the node admittance matrix.

Let $e_{q_1}, e_{q_2}, \ldots, e_{q_{e-v}}$ be a set of elements giving a non-zero major in B_{-1} and B'_{-2}. In order to establish the signs of these two majors the complete f-circuit matrix B_f is considered in which the columns are rearranged in the order

$$1, 2\ q_1, q_2, \ldots, q_{e-v}, \ldots, q_{e-2}.$$

Since the order of the columns $q_1, q_2, \ldots, q_{e-v}$ has not been changed, the major determinants of interest remain the same. Now the set of elements complementary to the set

$$q_1, q_2, \ldots, q_{e-v}$$

(with respect to N) is a 2-tree of N separating the pairs of vertices $(1, 1')$ and $(2, 2')$. If both E and Z_L are adjoined to this 2-tree, the resultant graph contains one circuit K containing both E and Z_L. Since every circuit can be built up from fundamental circuits, so can K. Let the coefficients of the linear combination of fundamental circuits, which produces K, be

$(\epsilon_1, \epsilon_2, \ldots, \epsilon_\mu)$;
where each $\epsilon_j = 1,\ -1$ or 0, $\mu = e - v + 1$.

Since $\epsilon_1 = 1$, then $\epsilon_2 = 1$ or -1. (Since K contains both E and Z_L and these appear only in the first and second circuits of the fundamental set, respectively, $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$.) Therefore

$$[\epsilon_1\ \epsilon_2\ \epsilon_3\ \ldots \epsilon_\mu] B_f = K \tag{186}$$

where K stands for the row matrix of the circuit K.

The circuit matrix B_f can be premultiplied by the non-singular matrix

$$M = \begin{bmatrix} 1 & 0 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \ldots & \epsilon_{\mu-1} & \epsilon_\mu \\ 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \tag{187}$$

Since rows 1 and 2 have not been used as 'tool' rows in this set of row operations, the major determinants of B_{-1} and B_{-2} are unaltered in this process.

Let
$$M \cdot B_f = B_K \tag{188}$$

In the matrix B_K if row 2 is multiplied by ϵ_2 and added to the first row, the first row becomes the circuit K. This circuit K contains E, Z_L and elements from the 2-tree. Therefore K does not contain any of the elements

$$q_1, q_2, \ldots, q_{e-v}.$$

Since $\epsilon_2 = 1$ or -1, the entries in columns $q_1, q_2, \ldots, q_{e-v}$ of the first two rows of the matrix B_K are either identical or the entries in the second row are the negatives of the entries in the first row.
Case (i): The entries in columns $q_1, q_2, \ldots, q_{e-v}$ of first two rows are identical.

Then the majors of B_{-1} and B_{-2} are identical since deleting the first row of the submatrix containing columns $q_1, \ldots, q_{e-v}$ produces the same submatrix as deleting the second row.

Therefore the product of the two majors is equal to one.

In this case $\epsilon_2 = -1$ producing the desired zeros for circuit K. Thus circuit K has the form

$$\begin{matrix} E & Z_L & q_1 & q_2 & q_3 & \ldots & q_{e-v} & . & . & . & . \\ [1 & -1 & 0 & 0 & 0 & \ldots & 0 & - & - & - & -]. \end{matrix}$$

E and Z_L appear with opposite signs. Referring to Fig. 18, the circuit K goes through the vertices of E and Z_L in the order

$$1'\,1\,2'\,2\,1'.$$

Therefore the 2-tree must be

$$T_{2_{12'},\,1'2}$$

to provide the required paths for circuit K of this form. The converse is also true.

Thus in the expansion of det $B_{-1}Z_eB'_{-2}$, $C[W_{12',\,1'2}(Z)]$ has a positive sign.
Case (ii): The entries of the first two rows of B_K in columns $q_1, \ldots, q_{e-v}$ have opposite signs.

In this case the major of B_{-2} is obtained by multiplying the first row of the major of B_{-1} by -1, so (major of B_{-1})X(major of B_{-2}) $= -1$. (189) Also in this case $\epsilon_2 = 1$, so following the argument used for circuit K

$$\begin{matrix} E & Z_L & q_1 & q_2 & \ldots & q_{e-v} & . & . & . & . & . & . \\ [1 & 1 & 0 & 0 & \ldots & 0 & - & - & - & - & - & -] \end{matrix}$$

Therefore the 2-tree must be of the type

$$T_{2_{12},\,1'2'}$$

to provide the required path for the circuit K. Conversely, every such 2-tree leads to a circuit K for which $\epsilon_2 = 1$.

In the expansion of det $B_{-1}\cdot Z_e\cdot B'_{-2}$, the sum $C[W_{12,\,1'2'}(Z)]$ has a negative sign.

Thus,

$$\det B_{-1}\cdot Z_e\cdot B'_{-2} = C\,[W_{12',\,1'2}\,(Z)] - C\,[W_{12,\,1'2'}\,(Z)]. \quad (190)$$

Finally since

$$\Delta_{12} = (-1)^{1+2}\det B_{-1}\cdot Z_e\cdot B'_{-2} = -\det B_{-1}\cdot Z_e\cdot B'_{-2} \quad (191)$$

the result is

$$\Delta_{12} = C\,[W_{12,\,1'2'}\,(Z)] - C\,[W_{12',\,1'2}\,(Z)]. \quad (192)$$

This formula can be written in an intuitive fashion following Percival as in Fig. 27.

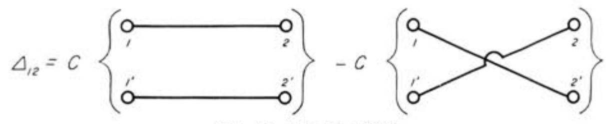


Fig. 27 and Eq. 193.

T 18: For a network containing no mutual inductances, if circuits 1 and 2 contain the elements $(1, 1')$ and $(2, 2')$ respectively and these elements are in no other circuits, the cofactor $(1, 2)$ of the mesh impedance matrix is given by

$$\Delta_{12} = C[W_{12,\,1'2'}(Z)] - C[W_{12',\,1'2}(Z)]. \quad (194)$$

23. Open Circuit Functions

In order to establish the topological formulas for the open circuit functions

$$Z_{oc} = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \quad (195)$$

(which is the coefficient matrix of the two terminal-pair equation

$$V_{2TP} = Z_{oc}\cdot I_{2TP}) \quad (196)$$

in terms of the impedances of the elements, it is more convenient to begin with the mesh equations of the network of Fig. 28.

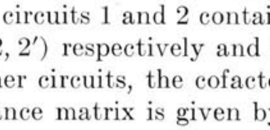


Fig. 28.

These equations have the form:

$$\begin{bmatrix} Z_{11} & Z_{12} & \ldots & Z_{1\mu} \\ Z_{12} & Z_{22} & \ldots & Z_{2\mu} \\ . & . & . & . \\ Z_{1\mu} & Z_{2\mu} & \ldots & Z_{\mu\mu} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \cdot\cdot \\ I_\mu \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \\ \cdot\cdot \\ 0 \end{bmatrix} \quad (197)$$

Solving this equation for I_1, I_2, since the right side contains only zeros after the second row, the result is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \Delta_{11}/\Delta & \Delta_{12}/\Delta \\ \Delta_{12}/\Delta & \Delta_{22}/\Delta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{198}$$

Finally inverting this equation:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \Delta_{22}/\Delta_{1122} & -\Delta_{12}/\Delta_{1122} \\ -\Delta_{12}/\Delta_{1122} & \Delta_{11}/\Delta_{1122} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{199}$$

which is the two terminal-pair equation desired. Comparing the entries in the coefficient matrix with topological formulas $T\,6$ and $T\,18$, it becomes apparent that:

$T\,19$: The open circuit functions of a network not containing any mutual inductances are given by

$$z_{11} = C[W_{1,1'}(Z)]/C[V(Z)] \tag{200}$$

$$z_{12} = z_{21} = \frac{C[W_{12',1'2}(Z)] - C[W_{12,1'2'}(Z)]}{C[V(Z)]} \tag{201}$$

$$z_{22} = C[W_{2,2'}(Z)]/C[V(Z)] \tag{202}$$

where V, W and the complement are with respect to the network N alone.

The result $\Delta_{1122} = C[V(Z)]$ follows because Δ_{1122} is simply the mesh determinant of the network (when elements V_1, V_2 are removed).

These formulas can be extended to give voltage and current ratio transfer functions by observing:

$$\mu_{12} = \left.\frac{V_2}{V_1}\right|_{I_2 = 0} = -\Delta_{12}/\Delta_{22} \tag{203}$$

$$\mu_{21} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0} = -\Delta_{12}/\Delta_{11} \tag{204}$$

$$\alpha_{12} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0} = \Delta_{12}/\Delta_{11} \tag{205}$$

$$\alpha_{21} = \left.\frac{I_1}{I_2}\right|_{V_1 = 0} = \Delta_{12}/\Delta_{22} \tag{206}$$

$T\,20$: The voltage and current ratio transfer functions of a two terminal-pair network not containing any mutual inductances are given by

$$\alpha_{12} = -\mu_{21} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{2,2'}(Z)] \tag{207}$$

$$\alpha_{21} = -\mu_{12} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{1,1'}(Z)] \tag{208}$$

Example

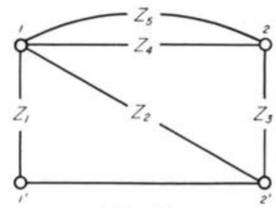


Fig. 29.

The open circuit functions for the graph of Fig. 29 are:

$$z_{11} = C[W_{1,1'}(Z)]/[V(Z)] \tag{209}$$

$$z_{12} = \{C[W_{12',1'2}(Z)] - C[W_{12,1'2'}(Z)]\}/C[V(Z)] \tag{210}$$

$$z_{22} = C[W_{2,2'}(Z)]/C[V(Z)] \tag{211}$$

$$V(Z) = Z_1Z_3 + Z_1Z_4 + Z_1Z_5 + Z_2Z_3 + Z_2Z_4 + Z_2Z_5 + Z_3Z_4 + Z_3Z_5. \tag{212}$$

$$C[V(Z)] = Z_2Z_4Z_5 + Z_2Z_3Z_5 + Z_2Z_3Z_4 + Z_1Z_4Z_5 + Z_1Z_3Z_5 + Z_1Z_3Z_4 + Z_1Z_2Z_5 + Z_1Z_2Z_4. \tag{213}$$

$$W_{2,2'}(Z) = Z_1 + Z_2 + Z_4 + Z_5 \tag{214}$$

$$C[W_{2,2'}(Z)] = Z_2Z_3Z_4Z_5 + Z_1Z_3Z_4Z_5 + Z_1Z_2Z_3Z_5 + Z_1Z_2Z_3Z_4. \tag{215}$$

$$W_{1,1'}(Z) = Z_3 + Z_4 + Z_5 \tag{216}$$

$$C[W_{1,1'}(Z)] = Z_1Z_2Z_4Z_5 + Z_1Z_2Z_3Z_5 + Z_1Z_2Z_3Z_4 \tag{217}$$

$$W_{12',1'2}(Z) = 0. \tag{218}$$

$$\text{Hence} \quad C[W_{12',1'2}(Z)] = 0 \text{ by convention.} \tag{219}$$

$$W_{12,1'2'}(Z) = Z_4 + Z_5 \tag{220}$$

$$C[W_{12,1'2'}(Z)] = Z_1Z_2Z_3Z_5 + Z_1Z_2Z_3Z_4. \tag{221}$$

Therefore the open circuit functions of the network are:

$$z_{22} = [Z_3Z_4Z_5(Z_1+Z_2) + Z_1Z_2Z_3(Z_4+Z_5)] / [Z_4Z_5(Z_1+Z_2) + Z_2Z_3(Z_4+Z_5) + Z_1Z_3(Z_4+Z_5) + Z_1Z_2(Z_4+Z_5)]. \tag{222}$$

$$z_{12} = [-Z_1Z_2Z_3(Z_4+Z_5)]/[Z_4Z_5(Z_1+Z_2) + (Z_2Z_3+Z_1Z_3+Z_1Z_2)(Z_4+Z_5)] \tag{223}$$

$$z_{11} = [Z_1Z_2(Z_3Z_4+Z_3Z_5+Z_4Z_5)]/[Z_4Z_5(Z_1+Z_2) + (Z_1Z_2+Z_1Z_3+Z_2Z_3)(Z_4+Z_5)] \tag{224}$$

$$\alpha_{21} = -\mu_{12} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{1,1'}(Z)] \tag{225}$$

$$\alpha_{21} = -\mu_{12} = Z_3(Z_4+Z_5)/[Z_3Z_4+Z_3Z_5+Z_4Z_5] \tag{226}$$

$$\alpha_{12} = -\mu_{21} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{2,2'}(Z)] \tag{227}$$

$$\alpha_{12} = -\mu_{21} = Z_1Z_2(Z_4+Z_5)/[Z_1Z_2(Z_4+Z_5) + Z_4Z_5(Z_1+Z_2)] \tag{228}$$

VII. APPLICATION OF TOPOLOGICAL FORMULAS TO NETWORK SYNTHESIS

24. One Terminal-Pair Networks

The application of topological formulas to the synthesis of driving point functions has been discussed at length by Seshu[22] and Kim.[14] Therefore this section is merely an outline of what has been done, and a statement of the unsolved problems.

It has already been shown in Chap. III that the driving point admittance of a one terminal-pair network is given by

$$Y_d = V(Y)/W(Y). \tag{229}$$

Synthesis normally begins with a positive real rational function of a complex variable, s,

$$Y_d(s) = p(s)/q(s) \tag{230}$$

where p and q are polynomials in s. In order to apply the topological methods the expression for $Y_d(s)$ must be converted into an expression in terms of $y_1, y_2, \ldots, y_e$. So that, a number of elementary positive real functions $y_1(s), y_2(s), \ldots, y_e(s)$ must be selected and $Y_d(s)$ expressed in terms of these. Just how these elementary positive real functions are to be chosen (if the final function is to be realizable as a network) remains still an unanswered question. Whitney[28] has stated the conditions that have to be satisfied by $V(Y)$, if it is to correspond to a matrix which has been extended[22] to a condition for a graph to exist. In other words, having chosen the functions and the combination, it is possible to check whether they are realizable. Any practical procedure would have to avoid the "cut and try" implied by this procedure.

In outline, the method of synthesizing a given ratio V/W is as follows:

(a) Examine V and W to see whether V can be written as

$$V = y_o W + V' \tag{231}$$

where y_o is one of the elementary functions and V' is free of y_o. If not, construct the function

$$V_1 = y_o W + V. \tag{232}$$

The rest of the procedure is for synthesizing V_1 in the latter case.

(b) Choose any product in V, containing y_o as the tree product, t_r. For convenience of notation, let

$$t_r = y_{e-v+2}\, y_{e-v+3} \ldots y_{e-1} y_o. \tag{233}$$

where e is the number of variables in V and $(v-1)$ is the degree of V.

(c) For each y_i not in this tree product, form π products π_{ri} as follows:

π_{ri} is the product of y_i by some or all of the elements in t_r, which is not a factor of any product in V.

If V is realizable, the π products for any y_i (for the chosen tree product) will contain a 'smallest' π product which is a factor of all others. Such a π product will be designated as the *f*-circuit product as it corresponds to the fundamental circuit.

(d) Construct the matrix B_f of these fundamental circuit products. B_f is of order $(e-v+1 \times e)$ with rows corresponding to *f*-circuits and columns corresponding to variables arranged in the order

$$y_1, y_2, \ldots, y_{e-v+1}, y_{e-v+2}, \ldots, y_{e-1}, y_o.$$

The matrix $B_f = [b_{ij}]$ is defined as

$b_{ij} = 1$ if y_j appears in the *f*-circuit product of y_i

$b_{ij} = 0$ otherwise.

With this definition the *f*-circuit matrix has the form

$$B_f = [U \;\; B_{12}]. \tag{234}$$

(e) Construct the fundamental cut set matrix

$$C_f = [B'_{12} \;\; U]. \tag{235}$$

(f) Reduce C_f by elementary row operations (modulo 2 operations) to a matrix in which each column has either 1 or 2 non-zero entries. This new matrix A is the incidence matrix of the graph of the network. The input vertices are the vertices of y_o. If y_o was added in step (a), delete y_o after marking its vertices as 1, 1′.

If W is the associated denominator polynomial and if B_{f1} is constructed for W, choosing the same tree product as before except for y_o, B_{f1} and B_f are identical except that column y_o is absent in B_{f1}.

Details and examples are in the original paper.[22]

Kim[14] introduced a variation on this procedure making it more powerful. Instead of leaving the realizability question until the end, Kim starts with a graph and specified types of elements, with the element values unknown. Using the topological formula T 5 Kim reduces this problem (by equating coefficients) to one of solving simultaneous multilinear equations. Kim derives a number of conditions for the realizability of a given function as a bridge network. The original report[14] gives the details of Kim's procedure. The two major points of interest in Kim's procedure are: (a) use of topological formulas for the synthesis of a one terminal-pair network, and (b) for a large class of functions the bridge circuit uses fewer elements than the classical procedures, including Brune's. The main drawback of the topological synthesis procedures for one terminal-pair networks is the difficulty of choosing elementary functions y_1, y_2, $\ldots$, y_e and polynomials V, W in such a manner that

$$Y_d = V/W \tag{236}$$

and V, W are realizable. One method of overcoming this difficulty would be to classify positive real functions in some manner which would guide the design engineer in his choice of elementary positive real functions and polynomials, V and W.

25. Synthesis of Two Terminal-Pair Networks — Problem Statement

The synthesis of two terminal-pair networks as presented begins with the open circuit functions. The open circuit impedance matrix Z_{oc} is, as before, the coefficient matrix of the two terminal-pair equations

$$V_{2TP} = Z_{oc} I_{2TP}. \tag{237}$$

Since it is more convenient to synthesize in terms of admittances rather than impedances, Z_{oc} is expressed in terms of the node equations (see Section 9):

$$Z_{oc} = \begin{bmatrix} \Delta_{11}/\Delta & (\Delta_{12}-\Delta_{12'})/\Delta \\ (\Delta_{12}-\Delta_{12'})/\Delta & (\Delta_{22}+\Delta_{2'2'}-2\Delta_{22'})/\Delta \end{bmatrix} \tag{238}$$

In terms of 2-trees the matrix is given by (see Section 9):

$$Z_{oc} = \begin{bmatrix} W_{1,1'}(Y)/V(Y) & \{W_{12,1'2'}(Y)-W_{12',1'2}(Y)\}/V(Y) \\ \{W_{12,1'2'}(Y)-W_{12',1'2}(Y)\}/V(Y) & W_{2,2'}(Y)/V(Y) \end{bmatrix} \tag{239}$$

And also

$$\begin{aligned} \det Z_{oc} &= \frac{W_{1,1'}(Y)\cdot W_{2,2'}(Y) - [W_{12,1'2'}(Y)-W_{12',1'2}(Y)]^2}{V(Y)^2} \\ &= \frac{U_{12',2,1'}+U_{1,2,1'2'}+U_{12,2',1'}+U_{1,2',1'2}}{V(Y)} \\ &= (\Sigma U)/V \end{aligned} \tag{240}$$

using the same notation as before.

In synthesizing two terminal-pair networks from the open circuit impedance matrix Z_{oc}, using topological methods, the given functions are expressed in terms of element admittance functions $y_1, y_2, \ldots, y_e$. The problem of choosing these functions remains unsolved. Starting with the polynomials

V, $W_{1,1'}$, $W_{2,2'}$ and $[W_{12,1'2'}-W_{12',1'2}]$ and proceeding with the synthesis, the first question to be answered is realizability.

26. Input and Output Elements

In order to discuss the realizability and in order to be able to find the input and output terminals after synthesis, an element y_o is needed across the input terminals and an element y_L across the output terminals. Procedures are necessary for checking whether the given functions contain such y_o and y_L and for introducing these elements if the original functions did not contain them.

If there is an element y_o across the vertices $(1, 1')$, $V(Y)$ can be expressed as

$$V(Y) = y_o W_{1,1'}(Y) + V_1(Y) \tag{241}$$

where V_1 does not contain y_o (see Seshu[22]).

Similarly if there is an element y_L across the vertices $(2, 2')$ $V(Y)$ can be expressed as

$$V(Y) = y_L W_{2,2'}(Y) + V_2(Y) \tag{242}$$

where $V_2(Y)$ does not contain y_L.

Thus it is quite easy to check for the presence of these elements. Next a procedure is introduced for adding the elements y_o and y_L if there were no such terms initially.

Addition of y_o and y_L modifies $V(Y)$ as

$$V'(Y) = V(Y) + y_o W_{1,1'} + y_L W_{2,2'} + y_o \cdot y_L(\Sigma U) \quad (243)$$

where V' is the new network determinant. This formula follows on observing that: (a) all the trees of the original network are trees of the new network; (b) any tree of the new network which contains y_o but not y_L becomes a 2-tree separating (1, 1′) when y_o is removed; (c) any tree containing y_L but not y_o is similarly a 2-tree (2, 2′) when y_L is removed; (d) any tree of the new network which contains both elements becomes a 3-tree when these are removed and in this 3-tree the pairs of vertices (1, 1′) and (2, 2′) are in different parts.

It is therefore necessary to find ΣU from the functions that are given. From Section 23,

$$\Sigma U = \{W_{1,1'} \cdot W_{2,2'} - [W_{12,1'2'} - W_{12',1'2}]^2\}/V(Y) \quad (244)$$

Thus ΣU can be found from the given functions.

The addition of y_o modifies $W_{2,2'}$ as

$$W'_{2,2'} = W_{2,2'} + y_o(\Sigma U) \quad (245)$$

(y_L does not change $W_{2,2'}$); and the addition of y_L modifies $W_{1,1'}$ as

$$W'_{1,1'} = W_{1,1'} + y_L(\Sigma U), \quad (246)$$

(y_o does not change $W_{1,1'}$).

The others are unaltered. Thus the corresponding set of functions can be derived for a network containing both y_o and y_L.

In the following both y_o and y_L are assumed to be present in the chosen functions.

27. Necessary Conditions for Realizability

In order to discuss the realizability criteria the fundamental circuit matrices must be constructed for the polynomials V, $W_{1,1'}$ and $W_{2,2'}$ by the procedure outlined in Section 23. As a tree product of V, a product is chosen which contains both y_o and y_L. This will always be possible except in the trivial case where $z_{12}=0$ which is excluded from this discussion. Let B_f be the fundamental circuit matrix for V with elements y_o and y_L as branches. Then, if y_o is omitted from this tree product of V, the tree product is in $W_{1,1'}$. Let B_{fo} be the circuit matrix for $W_{1,1'}$ for this tree product. Similarly if y_L is omitted from the tree product of V that has been chosen, the tree product is in $W_{2,2'}$. Let B_{fL} be the fundamental circuit matrix for $W_{2,2'}$ for this tree product.

The necessary conditions for the realizability of V, $W_{1,1'}$ and $W_{2,2'}$ can be now stated as:

1. B_{fo} is obtained by deleting column y_o from B_f.
2. B_{fL} is obtained by deleting column y_L from B_f.
3. Complements of non-singular submatrices of B_f, B_{fo}, and B_{fL} correspond one to one to the products in the polynomials V, $W_{1,1'}$ and $W_{2,2'}$, respectively.

Together with the condition that V corresponds to a graph these conditions are also sufficient for the realizability of V, $W_{1,1'}$ and $W_{2,2'}$. (The detailed justification of these statements may be found elsewhere.[22])

Thus the polynomial V by itself completely determines the network (to within a 2-isomorphism). The polynomials $W_{1,1'}$ and $W_{2,2'}$ help in finding the input and output vertices. Thus the network is completely fixed. If $[W_{12,1'2'} - W_{12',1'2}]$ is to be the corresponding transfer set of 2-trees, they have to satisfy strict conditions.

The first necessary condition to be satisfied is to get ΣU from the given functions, as a polynomial. Thus

1. $V(Y)$ must be a factor of $W_{1,1'}(Y) \cdot W_{2,2'}(Y) - [W_{12,1'2'}(Y) - W_{12',1'2}(Y)]^2$.

The second necessary condition is derived by observing that

$$W_{1,1'} = W_{12,1'2'} + W_{12',1'2} + W_{122',1} + W_{1,1'22'} \quad (247)$$

$$W_{2,2'} = W_{12,1'2'} + W_{12',1'2} + W_{11'2,2'} + W_{2,11'2'}. \quad (248)$$

2. The admittance products in $W_{12',1'2} - W_{12',12'}$ are precisely the products which appear in both $W_{1,1'}$ and $W_{2,2'}$.

28. An Unsolved Problem

The necessary conditions have been found to be satisfied by the terms in $(W_{12,1'2'} - W_{12',1'2})$, namely, that these are the common terms of $W_{1,1'}$ and $W_{2,2'}$. However, at this time, which of these products have a positive sign in $(W_{12,1'2'} - W_{12',1'2})$ and which have a negative sign can not be specified. It is to be noted that the same network realizes also the negative of the given transfer function if the labels of the output vertices are interchanged. Thus the problem is to find the products which are grouped together (and will therefore carry the same sign). The case where there are no negative products (or no positive products) is particularly simple and corresponds to the case of two terminal-pair networks with one common terminal.

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APPENDIX: TOPOLOGICAL FORMULAS

$T\,1$: For a network that contains no mutual inductances, the node determinant is given by

$$\Delta_n = \det Y_n(s) = \Sigma \text{ Tree Products} = V(Y).$$

$T\,2$: For a network that contains no mutual inductances the mesh determinant is given by

$$\Delta_m = \det Z_m(s) = 2^{2i}\Sigma \text{ Chord Set Products} = 2^{2i}\,C[V(Z)].$$

$T\,2'$: For the fundamental system of circuits of a network which contains no mutual inductances, the mesh determinant is given by

$$\Delta_m = C[V(Z)].$$

$T\,3$: For a network without mutual inductances, with fundamental circuits,

$$\Delta_m = (\det Z_e)\cdot\Delta_n.$$

$T\,4$: If r is the reference vertex of node equations, the cofactor of an element in the (i, i) position is given by

$$\Delta_{ii} = \Sigma T_{2_{i,r}} \text{ Products} = W_{i,r}(Y).$$

$T\,5$: For a network without mutual inductances, the driving point admittance at vertices $(1, 1')$ is given by

$$Y_d(s) = V(Y)/W_{1,1'}(Y).$$

$T\,6$: The cofactor of an element in the (i, i) position of the mesh impedance matrix of a network not containing mutuals is given by

$$\Delta_{ii} = C[V_1(Z)]$$

where V_1 is the tree polynomial when the (i, i) element is deleted.

$T\,7$: For a network without mutual inductances, the driving point impedance at vertices $(1, 1')$ is given by

$$Z_d(s) = C[W_{1,1'}(Z)]/C[V(Z)].$$

$T\,8$: Let $1'$ be the reference vertex of a system of node equations for a network not containing mutual inductances; the cofactor of an element in the (i, j) position is given by

$$\Delta_{ij} = \Sigma T_{2_{ij,1'}} \text{ Products} = W_{ij,1'}(Y).$$

$T\,9$: If Y_n is the node admittance matrix of a network which does not contain any mutual inductances, with vertex $1'$ as reference,

$$\Delta_{12} - \Delta_{12'} = W_{12,1'2'}(Y) - W_{12',1'2}(Y).$$

$T\,10$: For a two terminal-pair network which contains no mutual inductances, the matrix of the short circuit admittance functions is given by

$$Y_{sc} = \frac{1}{\Sigma U}\begin{bmatrix} W_{2,2'} & W_{12',1'2} - W_{12,1'2'} \\ W_{12',1'2} - W_{12,1'2'} & W_{1,1'} \end{bmatrix}$$

$T\,11$: For a two terminal-pair network which contains no mutual inductances, the determinant of the short circuit admittance matrix is

$$\det Y_{sc} = V(Y)/\Sigma U(Y).$$

$T\,12$: For a two terminal-pair network which contains no mutual inductances, the open circuit impedance matrix is given by

$$Z_{sc} = \frac{1}{V(Y)} \cdot \begin{bmatrix} W_{1,1'}(Y) & W_{12,1'2'}(Y) - W_{12',1'2}(Y) \\ W_{12,1'2'}(Y) - W_{12',1'2}(Y) & W_{2,2'}(Y) \end{bmatrix}$$

$T\,13$: For a two terminal-pair network which contains no mutual inductances, the determinant of the open circuit impedance matrix is

$$\det Z_{sc} = \Sigma U(Y)/V(Y).$$

$T\,14$: The voltage ratio transfer functions of a two terminal-pair network without mutual inductances are given by

$$\mu_{12} = [W_{12,1'2'}(Y) - W_{12',1'2}(Y)]/W_{1,1'}(Y)$$
$$\mu_{21} = [W_{12,1'2'}(Y) - W_{12',1'2}(Y)]/W_{2,2'}(Y).$$

T 15: The current ratio transfer functions of a two terminal-pair network without mutual inductances are given by

$$\alpha_{12} = [W_{12',1'2}(Y) - W_{12,1'2'}(Y)]/W_{2,2'}(Y)$$

$$\alpha_{21} = [W_{12',1'2}(Y) - W_{12,1'2'}(Y)]/W_{1,1'}(Y).$$

T 16: The admittances of the *T* equivalent of a two terminal-pair network without mutual inductances are given by

$$Y_1 = V(Y)/[W_{122',1'}(Y) + W_{1,1',22'}(Y) + 2W_{12',1'2}(Y)]$$

$$Y_2 = V(Y)/[W_{12,1'2'}(Y) - W_{12',1'2}(Y)]$$

$$Y_3 = V(Y)/[W_{11'2,2'}(Y) + W_{2,11'2'}(Y) + 2W_{12',1'2}(Y)].$$

T 17: The admittances of the elements of the π equivalent of a two terminal-pair network without mutual inductances are given by

$$Y_a = [W_{11'2,2'}(Y) + W_{2,11'2'}(Y) + 2W_{12',1'2}(Y)]/\Sigma U(Y)$$

$$Y_b = [W_{12,1'2'}(Y) - W_{12',1'2}(Y)]/\Sigma U(Y)$$

$$Y_c = [W_{122',1'}(Y) + W_{1,1'22'}(Y) + 2W_{12',1'2}(Y)]/\Sigma U(Y)$$

T 18: For a network containing no mutual inductances, if circuits 1 and 2 contain the elements (1, 1′) and (2, 2′), respectively, and these elements are in no other circuits, the cofactor (1, 2) of the mesh impedance matrix is given by

$$\Delta_{12} = C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]$$

where the complementation is with respect to the network without the elements (1, 1′) and (2, 2′).

T 19: The open circuit functions of a two terminal-pair network which does not contain any mutual inductances, are given by

$$z_{11} = C[W_{1,1'}(Z)]/C[V(Z)]$$

$$z_{12} = \{C[W_{12',1'2}(Z)] - C[W_{12,1'2'}(Z)]\}/C[V(Z)]$$

$$z_{22} = C[W_{2,2'}(Z)]/C[V(Z)].$$

T 20: The voltage and current ratio transfer functions of a two terminal-pair network not containing any mutual inductances are given by

$$\alpha_{12} = -\mu_{21} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{2,2'}(Z)]$$

$$\alpha_{21} = -\mu_{12} = \{C[W_{12,1'2'}(Z)] - C[W_{12',1'2}(Z)]\}/C[W_{1,1'}(Z)].$$

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