Quantum Entanglement Lecture 8 2006-11-12

Density matrix: a more general way to make probability statements about a system
classical definition of entropy, probabilities
entanglement and unentangled probabilities
how states change with time
H is called the Hamiltonian, it is Hermitian, and an observable, the energy of the system.

$$\frac{\partial|\psi\rangle}{\partial t} = \frac{-i\,H}{\hbar}|\psi\rangle$$ **governs how every quantum state evolves in time**

entropy is the measure of entanglement?

Prof. Leonard Susskind; videos on [Stanford on iTunes U]
[Susskind's Blog: Physics for Everyone]

Density matrix: a more general way to make probability statements about a system.

(1)Density matrix is the quantum analogy of the probability distribution: F = Σ Fi * Pi

(2) Tr P = 1   the trace of the density matrix (probabilities) adds up to one;

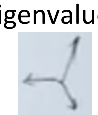
$$Tr\ \rho = 1$$

(3) The eignevalues of the Density Matrix can be though of as the probabilities of the different states;

(4) Density matrix is a Hermitian matrix (probabilities, the diagonal elements, are real);

$\lambda_i \qquad i = 1, \ldots N$ label the eigenvalues as λ

each eigenvalue corresponds to an eigen state vector;

 n mutually orthogonal basis vector of sub space

each eigenvalue corresponds to the probability that the system was prepared in the direction of that particular basis.

(5) minimum knowledge – all eigenvalues equal; maximum knowledge – only one eigenvalue > 0

$\rho = |\psi\rangle\langle\psi|$ the probability matrix is the projection onto that basis state.

09:30   with any pure state the dot product with that vector is the vector. any other vector the result is zero.

$$|\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$$

10:47   The average value (expectation) of the observable M is the trace*density matrix*M

$\bar{M} = Tr\ \rho M = \sum \langle i|\rho M|i\rangle$ $\bar{M} = Tr\ \rho\mathrm{M} = \sum_i \langle i|\rho \mathrm{M}|i\rangle$

13:39   example. for any M, in basis ψ (trace is the same in all basis) – summed over all indicies (basis)

$$M = \sum_i \langle i|\psi\rangle\langle\psi|M|i\rangle$$ M = Σ ⟨ i|ψ⟩⟨ψ|M|i⟩

$$\sum_i \langle\psi|M|i\rangle\langle i|\psi\rangle$$ Σ⟨ψ|M|i⟩⟨ i|ψ⟩ -- shifting <i|ψ>

14 :04 the expression Σ |i><i| is the unit matrix I which can be cancelled out, leaving

$$\langle\psi|M|\psi\rangle$$ the average of M is the expectation value of M (if a pure state)

16:19 General case, where there are mixed eigenvalues. you need to sum probabilities ρ against all indices in M:

$$\sum_{ij} \langle i|\rho|j\rangle\langle j|M|i\rangle$$

if I am working in the basis in which ρ is diagonal (i.e. probabilities on the diagonal) then I know what the matrix element <I|ρ|j> is. In order for their to be a value I must equal j (because ρ, the probability, is diagonal) so this is the sum over the eigenvalues λj

$$\sum_j \lambda_j \langle j|M|j\rangle$$ Σ λj ⟨j|M|j⟩ when ρ, the density matrix, has probabilities on the diagonal

17:46 <j|M|j> is the expectation value of M in the j^{th} state, λj is the probability. so the above expression is the average of M in the j^{th} state weighed with the probability of the j^{th} state.

the eigenvalues λ can be though of as the probabilities – but only in the case where ρ is diagonal.

19:56 classical definition of entropy

$$P_i \qquad S = -\sum_i P_i \log P_i$$

(minus because Pi ≤ 1 which means logPi is negative)
--- if only one Pi (=1) the entropy=0; complete knowledge of the system;
---if all Pi equal, entropy is maximum, complete ignorance.

22:00 entanglement

Say we have a system |a b> where a goes from 1 to N, and b goes from 1 to m. the total number of states for this system is N * m;

24:15 then the system is described as the sum, over a and b, of some function Ψ (the wave function) of a and b) – a nd b are discrete values (now, for our purposes) multiplied by the state vector a b

$$\sum_{ab} \Psi(a,b)|ab\rangle$$ Σ Ψ(a,b) |a b> (summed over a and b)

to normalize the sum Σ Ψ*Ψ , overa and b, must add to one.

25:30 experiment which only involves sub-system a, M = is observable for a

which means when M acts on a subsystem |a b>, M will act on the a coordinate but leave the b coordinate alone.

27:49 to determine the expectation of M on the combined system we need to construct a bra vector (introducing primed variables)

$$\sum_{a'b'} \psi^*(a'b')\langle a'b'|$$

against M and the ket; sum over a' b' to find bra vector; sum over a b to find the ket vector

$$\sum_{\substack{a'b'\\ ab}} \psi^*(a'b')\langle a'b'|M|ab\rangle \psi(ab)$$

29:00 M does nothing to b, so b' and b must be the same.

aside:

for state vectors, not density matrices expectation for a state vector is: say we have a wave function Ψ(a)|a> (simple system) and now we want to calculate the expectation of an observable M.

$$\sum_{a'} \psi(a)\langle a'|M \sum_{a} \psi(a)|a\rangle$$ $\sum_{a'} \psi(a)\langle a'|M \ \sum_a \psi(a)|a\rangle$

This is the value <s'|M|a>, where, in this case: (you have 2 sums to do)

31:00 b' b being the same means b is a diagonal and M is only a matrix of a' a

$$\psi^*(a'b)\, M_{a'a}\, \psi(ab)$$

ignore that the sum goes over a' a; and just do the sum over b

$$\sum_b \psi^*(a'b)\, M_{a'a}\, \psi(ab)$$

rearranging values that depend on b;

$$M_{a'a}\, \underbrace{\psi(ab)\psi^*(a'b)}$$
$$M_{a'a}\, \rho_{aa'}$$

we *call the expression Ψ(a b) Ψ*(a' b)* $\rho_{a\,a'}$

34:50 this function is the trace of M times P

$$\sum_{aa'} M_{a'a}\rho_{aa'} = Tr\, M\rho$$ $\sum_{aa'} M_{a'a}\rho_{aa'} \ = Tr\ M\rho$

36:00 checking the above identity

$\sum_{a'} \langle a'|M\rho|a'\rangle$ definition of a trace:
insert a complete set of states a |a><a|, a DYAD

$\sum_{aa'} \langle a'|M|a\rangle\langle a|\rho|a'\rangle$

<a'|M|a> is just Ma'a (M containing only a' a
<a|P|a'> is just P a a' (P containing only a a'

37:00 when you have a pure state of |a b> but you focus only on the a subsystem the system is described by a density matrix.

you obtain the density matrix by taking the full wave function and summing over b

$\Psi(ab)\Psi^*(a'b)$ sum over b: $\sum_b \psi(a\;b)\psi^*(a^{'}b)$

37:46 when you have a combined composite system |a b>, say Shrodinger's cat and a gun; then the cat and the gun is described by an entangled system but the cat alone is described by a density matrix.

in general, that density matrix will be of mixed phase. There is a specific case which it will describe a as a pure state.

38:50 the trace of a product is not the product of a trace (non-communative)
(Tr MP ≠ MP Tr)

40:20 you may have complete knowledge of the combined system Ψ(a b) – but when you select a sub-system a; you must describe that sub-system by a density matrix.
---in classical systems, if you have complete knowledge of a system you also have complete knowledge of the sub-systems.

42:30 is there a condition on Ψ(a b) so that the sub-systems are in pure states?
yes – anytime the wave function factorizes into products:

suppose: $\psi(ab) = \phi(a)\,\chi(b)$ ψ factors into φ(a) χ(b)

expand the density matrix Pa a'

$\rho_{aa'} = \sum_b \phi(a)\chi(b)\,\phi^*(a')\,\chi(b)$ $P_{a\,a'} = \sum_b \phi(a)\chi(b)\phi^*(a^{'})\chi^*(b)$

seperating Φ(a) – as sum does not depend on a;

$\phi(a)\phi^*(a')\sum_b \chi(b)\chi^*(b)$ $\phi(a)\phi^*(a^{'})\sum_b \chi(b)\chi^*(b)$

but $\sum_b \chi(b)\chi^*(b)$ is just one; so the density matrix of a product is just the product of wave functions Φ(a) Φ*(a)

$$P_{aa'} = \sum_b \phi(a)\chi(b)\phi^*(a')\chi^*(b)$$
$$= \phi(a)\phi^*(a')$$

calculate the expectation value of M with these Φ functions:

$\bar{M} = \langle \phi^*(a')|M_{a'a}|\phi(a)\rangle = \langle\phi|M|\phi\rangle$ which is exactly the same if the state vector was just Φ

Rule: When a pure system can be expressed by two factors of functions then each system can be described as a pure state of each function.
--- the density matrix has only one value, all the others are zero. (i.e. zero entropy)

49:00 One way of determining how close you are to a pure state is to calculate the entropy of the state. If the entropy is low you are close to a pure state.
--- deeply entangled means close to maximum entropy.
(calculate entropy by determing change increment over time)

50:00 Example. Calculate a Density Matrix and the Entropy

ex.1 highly entangled singlet state of 2 electrons:

$$\frac{|ud\rangle - |du\rangle}{\sqrt{2}}$$

write in terms of a 2 variable wave function

$\Psi(u\,u) = 0$ no u u component

$\Psi(ud) = \frac{1}{\sqrt{2}}$ value of u d component

$\Psi(d\,u) = -\frac{1}{\sqrt{2}}$ value of d u component

$\Psi(dd) = 0$ no d d component

calculate the density matrix of the a system; the a system being spin (of particle) #1

$$\Psi(a\,b)\Psi^*(a'b)$$

Puu is the a a' density matrix – start first with spin up (note a' is *)
we sum over b.

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calculation of the P (uu dd ud du) components:
calculate for P{u d}
set a = Ψu      Ψ*d   +  Ψu   Ψ*d
set b = Ψuu     Ψ*uu  +  Ψud  Ψ*dd
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Puu = Ψuu Ψ*uu + Ψud Ψ*ud
Pdd = Ψdu Ψ*du + Ψdd Ψ*dd
Pud = Ψuu Ψ*du + Ψud Ψ*dd

Pdu = Ψdu   Ψ*uu + Ψdd   Ψ*ud

$\rho_{uu} = \psi_{uu} \quad \psi^*_{uu} + \psi_{ud} \quad \psi^*_{ud}$ Puu = 0 × 0 + 1/√2 × 1/√2 = ½

$\rho_{dd} \quad \psi_{du} \quad \psi^*_{du} + \psi_{dd} \quad \psi^*_{dd}$ Pdd = - 1/√2 × -1/√2 + 0 × 0 = - ½

Pud = Pdu = 0

55:39 Density matrix is:
trace is 1; proportional to the unit matrix.
entropy is log 2 (1/2log1/2 + 1/2log1/2) = ½ (-log2) + ½ (-log2) = -log2 (but entropy has – sign in definition)
because the diagonal elements are both non-zero, and equal this means the singlet state is a non-pure.

$\rho = \begin{pmatrix} \frac12 & 0 \\ 0 & \frac12 \end{pmatrix}$ *the singlet state is a maximally mixed state*, maximum ignorance

57:00 what is the expectation value of spin #1
P = ½ (x unit matrix)
expectation value (along a σ matrix) is trace x P x sigma matrix
which is:

½ Tr σ.n

but trace of any sigma matrix is zero – so – expectation value is zero

59:00 in the singlet state, the spin of any component is equally likely to be along that direction or opposite to that direction.

$\rho = \frac{1}{2}$

$\overline{\sigma\cdot\hat{n}} = \mathrm{Tr}\ \rho\ \sigma\cdot n$

60:00 lets define a wave function as:

$\psi(uu) = \frac{1}{2}$

$\psi(ud) = \frac{1}{2}$

$\psi(du) = \frac{1}{2}$

$\psi(dd) = \frac{1}{2}$ a normalized state with all entries equal to ½

recalculating:
Puu = Ψuu   Ψ*uu + Ψud   Ψ*ud   = ½ x ½ + ½ x ½ = ½
Pdd = Ψdu   Ψ*du + Ψdd   Ψ*dd   = ½ x ½ + ½ x ½ = ½
Pud = Ψuu   Ψ*du + Ψud   Ψ*dd   = ½ x ½ + ½ x ½ = ½
Pdu = Ψdu   Ψ*uu + Ψdd   Ψ*ud   = ½ x ½ + ½ x ½ = ½

the density matrix is: $\rho = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

62:00 the trace is still equal to one, but the eigenvectors are different:
theorems:

1. *the product of the eigenvalues is equal to the determinant;*
2. *the sum of the eigenvalues is equal to the trace*

(in this case determinant = 0) so the eigenvalues are 1 0
corresponds to the product state of 2 unentangled electrons aligned along the x-axis ??? (why)

σ3 x P – the trace is zero; so the expectation value of σ3 is 0; same is true for σ2

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

σ3 x P; Tr=0 same result for σ2

65:00 expectation value for σ1 is 1; trace is equal to 1
entropy is zero, because it has one eigenvalue of 1, the other zero
a pair of spins which are both lying along the x-axis non-entangled

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

σ1 × P; Tr=1

66:00 if you slightly change the wave function, say Ψud = Ψdu = .4 you would find a small degree of entanglement, the entropy would have a small value. *Entropy and entanglement go together (called entanglement entropy)*

68:00 in general entropy does not add, in particular entanglement entropy.
a system may have an entropy of zero – but the sub-systems each may have entropy.

70:00 how states change with time

a discrete system goes from state to state, but not necessarily continuous.
i.e. the dimension of the vector space is finite, not continuous

classical – note 2 different states do not evolve to the same state.
but you can have a possibility of a continuous space.
The space {d u} can be measured as spin up or spin down but there can be a continuous number of states in between.

73:00
assumption: the logical relationships between states doesn't change with time;
(1)if 2 states are equal they will stay that way;
(2)if 2 states are orthogonal they will evolove into orthogonal states (if 2 states are measurably different they will remain measurably different)

in classical systems this is called conservation of phase state volume;
in quantum mechanics refers to unit parity.

77:00 **Second principle of time evolution**
what it means is the inner product between two states (magnitude) stays the same with time.

78:00 **first principle of time evolution.**
governed by a linear operator.
take a space |Ψ 0> (at time zero). after a while it evolves to a new state, |Ψ t>
|Ψ t> is always equal to U (some linear operator) times |Ψ 0>

$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$ ψ evolves linearly governed by an time oprator

note U(0)=1; |Ψ t> = U(0) |Ψ 0>

Principle 1: there exists a linear operator U(t) which describes the system evolution. (sort of an observed, empirical theory)
Principle 2: if there are any 2 systems that evolve in time – the dot product remains the same

$\langle\phi(t)|\psi(t)\rangle = \langle\phi(0)|\psi(0)\rangle$ dot product at time t equals dot product at time 0

assumption (repeated): The logical relations between states are invariant in time.

82:00 determine what happens to operator U (a matrix) in time; take the bra

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$
$$\langle\varphi(t)| = \langle\varphi(0)|\,U^\dagger(t)$$
$$\langle\phi(t)|\psi(t)\rangle = \langle\phi(0)|\psi(0)\rangle$$
$$\langle\varphi(0)|U^\dagger U|\psi(0)\rangle = \langle\phi(0)|\psi(0)\rangle$$

which makes the last relation as U^t U being the unit operator. U is a unitary operator.

$$U^\dagger U = 1$$

the time evolution of a system is governed by unitary operators that depend on time

85:00 lets take a very small interval of time ϵ, and define: U(ϵ) = 1 – I ϵ H

$U(\varepsilon) = 1 - i\varepsilon H$ minus sign arbitrary, as is i

87:00 determine the conditions on H

$U(\varepsilon) = 1 - i\varepsilon H$

$U^\dagger(\varepsilon) = 1 + i\varepsilon H^\dagger$

multiplying out, remember $U^T U = 1$

$(1 + i\varepsilon H^\dagger)(1 - i\varepsilon H) = 1$

$1 - i\varepsilon H + i\varepsilon H^\dagger = 1$

$i\varepsilon(H^\dagger - H) = 0$ ignoring ϵ squared term (very small)

or $H^T = H$, the H is Hermitian.

89:55 H is called the Hamiltonian, it is Hermitian, and an observable, the energy of the system.
The eigenvalues of the matrx H are the energy levels of the system.

$|\psi(\varepsilon)\rangle - |\psi(0)\rangle = -i\varepsilon H|\psi\rangle$

$(1 - i\varepsilon H)|\psi(0)\rangle - |\psi(0)\rangle$ small change in the system

$\frac{|\psi(\varepsilon)\rangle - |\psi(0)\rangle}{\varepsilon} = -i H|\psi\rangle$ rearranging ϵ , the small unit of time

$$\boxed{\frac{\partial|\psi\rangle}{\partial t} = -i\frac{H}{\hbar}|\psi\rangle}$$

93:00 hamiltonian

$\frac{\partial|\psi\rangle}{\partial t} = \frac{-i\,H}{\hbar}|\psi\rangle$ **governs how every quantum state evolves in time**.

98:00 entropy is the measure of entanglement?
entropy is dimensionless.

in classical thermodynamics:

$$dE = T\,dS$$

$$KE = \frac{3}{2}\,T$$

$$T = \frac{\varepsilon_m}{k}$$

the difference in energy = time x difference in entropy;
time has units of energy;
formula for KE contains Boltzman constant – which is only necessary to convert time from units of enegry to units of time.

100:00
say we have an eigenvector Ψ ; H hamiltonian, E (energy) is an eigenvector

$$H|\Psi\rangle = E|\Psi\rangle$$

$$|\Psi(t)\rangle = f(t)|\Psi(0)\rangle$$ equation for Ψ evolving with time;

substituting with $\partial|\Psi\rangle/\partial t = -\,i\,H|\Psi\rangle/\hbar$

$$\frac{\partial f}{\partial t} = -i\,\frac{E}{\hbar}\,f$$

$$f = e^{-i\frac{E}{\hbar}t}\,f(0)$$

$$\hbar\,\omega = E$$

if you start with a start with an eigenvector of the hamiltonian then it evolves with time just like multiplying with a phase.

the value (- I E/$\hbar$), when modifying time in the above, is called the angular frequency ω --- E = hλ