Quantum Entanglement Lecture 5 2006-10-23

review action on sigma matrices
the expectation value for all sigma observable directions is zero (50% up, 50% down)
which + or – is an eigenvector of the entangled state – the singlet state
Bell's Theorem (a classical probability theorem)
Bell's Theorem not true in entangled state
Calculate sigma projection operators
alternate definition of probability using projection operators
proof you cannot clone a quantum system

Prof. Leonard Susskind; videos on Stanford on iTunes U
Susskind's Blog: Physics for Everyone

03:00 review action on sigma matrices;

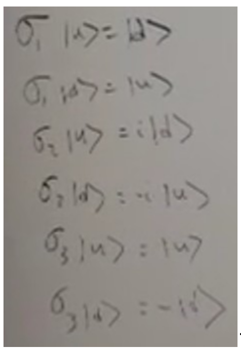
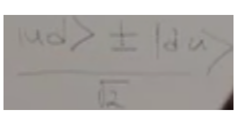
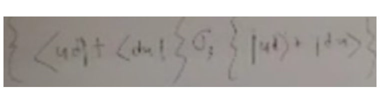


table of how sigma matrices affect up/down states

Sigma operations on up/down		
σ1|u>	+1|d>	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$
σ1|d>	+1|u>	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$
σ2|u>	+i|d>	$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\i\end{bmatrix}$
σ2|d>	-i|u>	$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-i\\0\end{bmatrix}$
σ3|u>	+1|u>	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$
σ3|d>	-1|d>	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$


entangled states of 2 electrons. if one is up the other is down

|u d> ± |d u> entangled states of 2 electrons


in either of these states the expectation value is zero (<a|M|a> = 0)

review expectation value:

a vector state of the system you wish to measure;

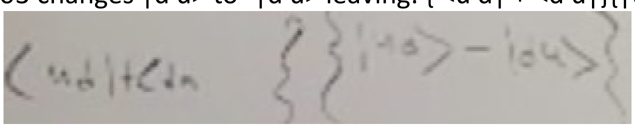
M matrix of observables;

<a|M|a> the measure of whether vector state a is observed at M

say we have <a|σ3|a> where <a| is the state “up” (<1 0|) and σ3 is the z-axis observable.
1-to measure whether an electron is in the “up” (<1 0|) state; place a magnet with “N” pointing “up along the z-axis”.
2-if an photon (i.e. energy) is emitted the electron has moved to the “up along the z-axis” state;
3-<a|σ3|a> is a measure of the expected or average value; $|\langle a|\sigma 3|a\rangle|^2$ is the probability.

{< u d| + < d u|} σ3 {|u d> + |d u>}

σ3 changes |d u> to -|d u> leaving: { <u d| + <d u|}{|u d> - |d u>}



mismatched terms are zero;
<u d|u d> = 1; <d u| - d u> is -1 ; sum is zero

in an entangled state the expectation value of any sigma operator is 0

assume a 2 electron entangled state and are measuring the spin “up” or “down”. with the entanglement when one electron is in the “up” state the other will be in the “down” state. Remember the “up”,”down” states do not refer to any absolute direction but refer to where the direction “North” of the magnet.

measuring “up” is a combined state of electron 1 being “up” or electron 2 being “up”;

a = {u d} + {d u}

rotate our z-axis so that z+ is the magnets north – and consequently use sigma3 as the observable. Note we have 2 observables which we will call σ3 for electron 1 and τ3 for electron 2.

The bra ket equation for the expected or average value of measurable 1 is:

{<u d| + <d u|} σ3 {|u d> + |d u>}

we apply the σ3 operator to the right term. Check table above remembering σ3 only operates on the first electron, i.e. σ3|u d> is |u d> and σ3|d u> is minus |d u>:

{<u d| + <d u|} {|u d> - |d u>}

expanding this we get 4 terms (to be added together;

(1)<u d|u d> = 1 dot product of 2 identical “unit” vectors;
(2) -<u d|d u> = 0 vectors are orthogonal, dot product is zero;
(3)<d u|u d> = 0 vectors are orthogonal, dot product is zero;
(4) -<d u|d u> = -1 dot product of 2 identical “unit” vectors

the sum is zero, therefore the expected or average value is zero.
This means that the expected value of *one electron* in a *two electron entangled* state is always “50% up” or “50% down”.

expansion in matrx or linear algebra form:

$$\{\langle(1\quad 0)(0\quad 1)| + \langle(0\quad 1)(1\quad 0)|\}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\{\left|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle + |\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\rangle\}$$

appplying operator on the first electron, right hand term:

$$\{\langle(1\quad 0)(0\quad 1)| + \langle(0\quad 1)(1\quad 0)|\}\{\left|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle - |\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\rangle\}$$

expansion: (terms to be added, split because they overflow the page width)

1 $\langle(1\quad 0)(0\quad 1)|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\rangle$

2 - $\langle(1\quad 0)(0\quad 1)|\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\rangle$

3 + $\langle(0\quad 1)(1\quad 0)|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\rangle$

4 - $\langle(0\quad 1)(1\quad 0)|\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\rangle$

calculating the dot products. not quite linear algebra – we apply <bra|ket> for each electron separately:

= (1*1+0*0)(0*0+1*1) – (1*0 + 0*1)(0*1 + 1*0) + (0*1 + 1*0)(1*0 + 0*1) – (0*0 + 1*1)(1*1 + 0*0)

= 1 – 0 + 0 – 1 = 0

The Dirac <bra|ket> equation is identical but much easier than linear algebra

the expectation value for all sigma observable directions is zero (50% up, 50% down)

substitute any sigma observable and you get the sam result,

i.e. use τ2 (2nd electron along y-axis) $\tau 2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

{<u d| + <d u|} τ2 {|u d> + |d u>}

{<u d| + <d u|} {-i |u d> + i |d u>}

you get the result (-i + 0 + 0 + i) = 0

14:00 which + or – is an eigenvector of the entangled state – the singlet state

we want to find which state of the entangled system (|u d> ± |d u>) leads to an eigenvector, i.e. M|a> = λ|a>

test the expectation value for any combination of operators (σ? + τ?)

(σ? + τ?)(|u d> - |d u>) always gives zero;

(σ? + τ?)(|u d> + |d u>) does not.

This means that (|u d> - |d u>) (with the minus sign) has an expectation value of zero – and is an eigenvalue of the entangled operator (σ? + τ?). thus: (|u d> - |d u>) is called the singlet state, an eigen vector. The state (|u d> + |d u>) is called the triplet state.

singlet state: |u d> - |d u>

a **singlet** usually refers to a one-dimensional representation (e.g. a particle with vanishing spin. All σ correlations are zero.

(say, σ3 operates on one electron; τ3 operates on the other):

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we split the operator {σ3+τ3} as per commutative rule:
      {σ3+τ3}{|u d>-|d u>}=σ3{|u d>-|d u>} + τ3{|u d>-|d u>}

σ3{|u d>-|d u>} + τ3{|u d>-|d u>}={|u d>+|d u>} + {-|u d>-|d u>}=0
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σ1{|u d>-|d u>} + τ1{|u d>-|d u>}={|d d>+|u u>} + {-|u u> -|d d>}=0
σ2{|u d>-|d u>} + τ2{|u d>-|d u>}={-i|d d>+i|u u>}+{-i|u u>+i|d d>}=0
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also, any vector spin zero as all σ components are zero: {σ2 + τ2}{n} = 0

triplet state: |u d> + |d u>

calculate, as above the {σ1 + τ1}{ |u d> + |d u>}

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σ1{|u d>+|d u>} + τ1{|u d>+|d u>}={|d d>-|u u>}+{-|u u> +|d d>}=2|d d>-1|u u>
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28:00 Bell's Theorem (a classical probability theorem)

QM QE violates Bell's Theorem

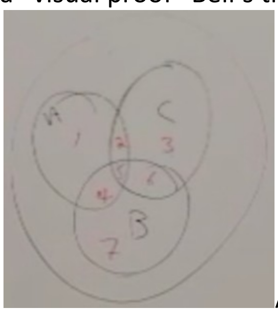
30:00

Any classical theory obeys set theory logic. given 3 states (A,B,C) then the classical set eq. is true.

$N(A\cap\sim B) + N(B\cap\sim C) \geq N(A\cap\sim C)$

the number of elements in (A and not B) plus the number of elements …

a "visual proof" Bell's theory holds for classical set theory



A,B,C enclosing 3 regions,

N1,N2,etc. are number of elements in labelled regions

A⋂~B = N1 + N2
B⋂~C = N7 + N4
A⋂~C = N1 + N4

$N(A\cap\sim B) + N(B\cap\sim C) \geq N(A\cap\sim C)$

N1 + N2 + N4 + N7 ≥ N1 + N4 (equal only if N2+N7=0, otherwise greater)

In classical probabilities, N (number) corresponds to number of elements in the selected set.

If M3=N{A⋃B⋃C}; N(M3 ⋂(A⋂~B)) + N(M3 ⋂(B⋂~C))> N(M3 ⋂(A⋂~C)), ≥ N(B⋂C)=0

In quantum probabilities, N corresponds to vector space.

38:00 Bell's Theorem not true in entangled state

Take 2 electrons, singlet state, 1 spin only.

A = 1up along 0° angle z–axis

B = 1up along 45° angle

C = 1up along 90° angle

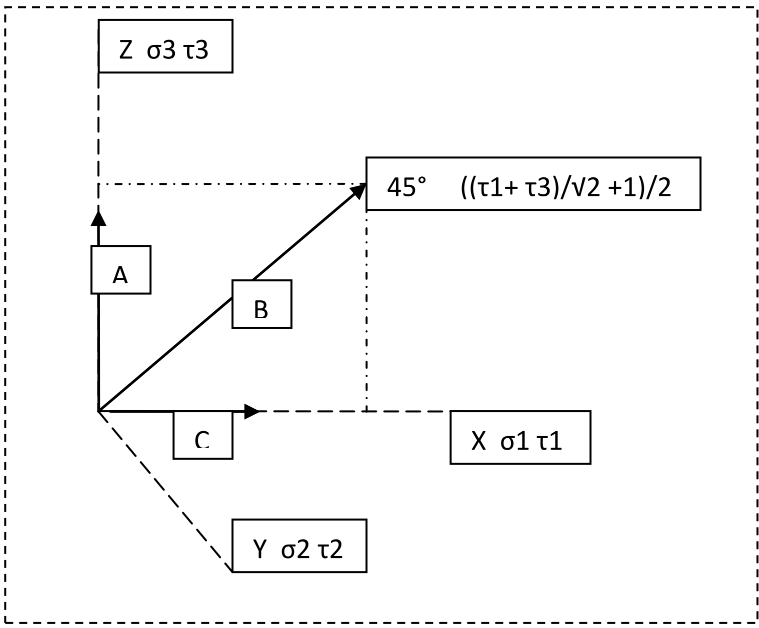
~B => 2up along 45°

~C => 2up along 90°

(i.e. other spin in opposite direction). That is, the negative of one electron spin direction is the positive of the other electron spin:

i.e. if 1st is down, then 2nd is up.

this allows us to always express a "down" state of one electron as an "up" state of the other.



vector observable (spin) measured at A,B,C states defn. above

Note: the sigma operator for electron 2, up45° is (τ1 + τ1) – normalized by (1/√2). The value on the chart ((τ1+ τ3)/√2 +1)/2 is the projection operator for 2, up45° . (√2 hypotenus side of unit vector 45° triangle)

40:00 reform Bell's theorem:

$N(A \cap \sim B) + N(B \cap \sim C) \geq N(A \cap \sim C)$

$N(A \cap \sim B) = (A, \sim B) = N(1up0°, 2up45°)$ ~B is the same as 2nd electron up at 45°

$N(B \cap \sim C) = (B, \sim C) = N(1up45°, 2up90°)$ ~C is the same as 2nd electron up at 90°

as previously discussed the probability is not affected by rotation. The state (B, ~C) is the same as (A, ~B) under rotation of 45° so our equation becomes:

$2 * N(1up0°, 2up45°) \geq N(1up0°, 2up90°)$

45:00 Calculate sigma projection operators

Require projection operators to calculate probability (i.e. Number) of N(1up0°,2up45°). vectors are orthogonal so operator is (1 0)

defn: a Projection operator projects a vector in n+m vector space to the n vector sub-space. i.e. given a vector a={a1 a2 a3) in 3D space project to 2D space:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} a1 \\ a2 \\ a3 \end{pmatrix} = \begin{pmatrix} a1 \\ a2 \\ 0 \end{pmatrix}$$

$\mathbb{P}_{12}$|a> = |a_{12}>

we are looking for a projection that projects vectors of observables (i.e. spin) to the sigma axis.

$\mathbb{P}$n =projection operator for σ_n in the "up" state
$\mathbb{P}_{\sigma3+1}$ = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ operator to project "up along z-axis" state.
given a state vector |α β> the projection to σ3 is $\mathbb{P}_{\sigma3+1}$|α β> = |α 0>
we can write as $\mathbb{P}_{\sigma3+1}$, or simply $\mathbb{P}$3 in a more general form: $\mathbb{P}$3 = (σ3 + I)/2

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \div 2$$

the same is true for all sigma operators. The projection operator is defn. as:

$\mathbb{P}$n = |σn,+>< σn,+|

which is the product of the "up" eigenvector of σn with its adjoint:

$\mathbb{P}3 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \quad 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathbb{P}2 \qquad \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} (1 \quad -\mathrm{i}) = \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$

$\mathbb{P}1 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \quad 1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Note: the σ1 and σ1 eigenvectors need to be normalized by dividing by (1/√2), the matrix, of course by ½
.

calculating the (σn + I)/2 projection operators:

$\mathbb{P}2 \qquad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \div 2$

$\mathbb{P}1 \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \div 2$

55:00 alternate definition of probability using projection operators

<φ|$\mathbb{P}$ |φ> expectation value of the projection operator
if you have a state, φ, the probability of that state is the projection operator on that state.
---- the expectation value is the probability.

the probability of z+ state is

$$\langle \alpha\ \beta | \frac{\sigma3+I}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \langle \alpha\ \beta | \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \alpha^* \alpha$$

({τ1+τ3)√2 +1)/2 {α}
{β}

τ1[a b]=[b a]
τ3[a b]=[a -b]

57:00 summary of Bell's theorem, so far
-1- classical probabilities correspond to subset of a set
-2- quantum probabilities correspond to subset of a vector space

state formula:

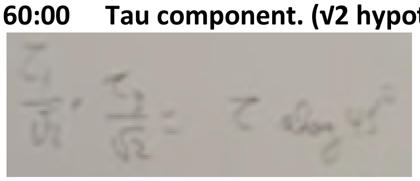
2* N(1up0°,2up45°) ≥ N(1up0°,2up90°)
(the (B,~C) same as (A,~B) under rotation of 45°)

2* N(1up0°,2up45°) is a product of individual projection operators
1—first take {(σ3 +1)/2} then multiply by {((τ1+τ3)/√2 + 1)/2}

we use the quantum singlet state of a pair of entangled electrons:

(|u d>-|d u>)/√2

60:00 Tau component. (√2 hypotenus side of unit vector 45° triangle)



63:00 this corresponds to Aα⋂~Bα or Aα⋂Bτ
σ term is projection operator for 90°

1up0° projection operator is (σ3 +1)/2

τ term is projection operator for 45°
2up45° projection operator is ((τ1+τ3)/√2 + 1)/2

τ spin operator is unit vector between 1 & 3
normalized (τ1+τ3)/√2
generalized as any operator (τn + 1)/2

||ud>-|du> is the singlet state
singlet state on the left is complex conjugated to get amplitude

$$\langle singlet| \ \frac{\sigma 3+I}{2} \ \left\{\frac{\tau 1+\tau 3}{2\sqrt{2}}+\ \frac{1}{2}\right\} \{ \frac{|u\ d\rangle - |d\ u\rangle}{\sqrt{2}} \}$$

Note: σ and τ components are multiplied (corresponds to ⋂)

64:00 calculations:
(1)σ3+1 {1 0} component kills |d u> state; which leaves:

$$\langle singlet| \frac{1}{\sqrt{2}} \left\{\frac{\tau 1+\tau 3}{2\sqrt{2}}+\ \frac{1}{2}\right\} |u\ d\rangle$$

note the τ operates on second entry d

τ1 flips down to up (d -> u); giving |u u>.
But on the left hand side there is no |u u> state so τ1 gives something completely orthogonal. so we can remove the τ1 operator,leaving:

$$\langle singlet| \frac{1}{\sqrt{2}} \left\{ \frac{\tau 3}{2\sqrt{2}} + \frac{1}{2} \right\} |u\ d\rangle$$

τ3 |u d> gives -1 ({1 -1} |u {0 1} >); leaving just a number:

$$\{\frac{\langle u\ d| - \langle d\ u|}{\sqrt{2}}\} \frac{1}{\sqrt{2}} \left\{ \frac{-1}{2\sqrt{2}} + \frac{1}{2} \right\} |u\ d\rangle$$

67:00 expanding the singlet state on the LH term
|d u> won't contribute because it is orthogonal to |u d>

inner product of <u d| with |d u> is 1
- (inner product of any unit vector with itself)

69:00 so we are left with just numbers. Note the LHS is double (2* N(1up0°,2up45°))

LHS= $$2 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \left\{ \frac{-1}{2\sqrt{2}} + \frac{1}{2} \right\} = 0.15$$

review: probability
{singlet state} {projection operator} {singlet state}
({<u d| - <d u|}/√2) {projection operator} ({<u d| - <d u|}/√2)

69:00 calculate R.H.Side: N(1up0°,2up90°)
1up0° is σ3 on electron one; (+ on z-axis)
2up90° is τ1 on electron two; (+ on x-axis)

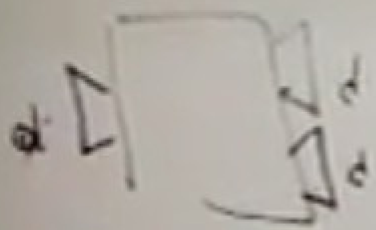
$$\left\{ \frac{\langle u\ d| - \langle d\ u|}{\sqrt{2}} \right\} \frac{\sigma 3 + I}{2} \frac{\tau 1 + I}{2} \left\{ \frac{|u\ d\rangle - |d\ u\rangle}{\sqrt{2}} \right\}$$

applying the operator to the RH term:
-1- σ3+1 {1 0} term gets rid of |d u> and leaves |u d> as-is;
-2- τ1 changes |u d> to |u u> but there is no |u u> on LH term, so that term is eliminated
leaving, again just numbers:

RHS= $$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = 0.25$$

But obviously .15 is not ≥ .25; so Bell's inequality is violated.

89:00 proof you cannot clone a quantum system
---no notes, best to listen



$$|u\rangle \longrightarrow |u\rangle|u\rangle$$

$$|d\rangle \longrightarrow |d\rangle|d\rangle$$

$$|r\rangle \longrightarrow |r\rangle|r\rangle$$

$$\frac{|u\rangle + |d\rangle}{\sqrt{2}} = \frac{|u\rangle|u\rangle + |d\rangle|d\rangle}{\sqrt{2}}$$

$$\left(\frac{|u\rangle + |d\rangle}{\sqrt{2}}\right)\left(\frac{|u\rangle + |d\rangle}{\sqrt{2}}\right)$$