Quantum Entanglement Lecture 3 2006-10-09

Quantum Mechanics is calculation of probabilities
sigma matrices
observables, eigenvalues and eigenvectors are orthogonal
can measure component of electron spin in any x,y,z
example, unit pointer in any arbitrary direction

Prof. Leonard Susskind; videos on [Stanford on iTunes U]
[Susskind's Blog: Physics for Everyone]

01:00 Quantum Mechanics is calculation of probabilities

|a> = α|up> + β|down>

α β are probabilities $\alpha^2 + \beta^2 = 1$

Hermitean matrices, some identities

- Transpose and complex conjugate, M_ij = M_ji *
- M† (dagger † is Hermitian conjugate)
- Real elements on diagonal,
- Complex conjugate off-diagonal

<b|a> = <a|b>* = $[b1 \quad b2]\begin{bmatrix} a1 \\ a2 \end{bmatrix} = \left([a1 \quad a2]\begin{bmatrix} b1 \\ b2 \end{bmatrix}\right)^*$

<b|a> = b1*a1 + b2*a2
<a|b> = a1*b1 + a2*b2
<a|b>* = b1*a1 + b2*a2 = <b|a>

<bra| is row vector, complex conjugate; |ket> is normal column vector

Hermitian calculation:

<b|M|a> = <a|M^T|b>* = $[b1 \quad b2]\begin{vmatrix} m11 & m12 \\ m12* & m22 \end{vmatrix}\begin{bmatrix} a1 \\ a2 \end{bmatrix} = \left([a1 \quad a2]\begin{vmatrix} m11 & m12 \\ m12* & m22 \end{vmatrix}\begin{bmatrix} b1 \\ b2 \end{bmatrix}\right)^*$

could multiply this out but as we know M = M† and above result it appears obvious.

- a is real
- expectation value of a

36:00 any observable can be represented as a collection of real numbers

39:00 eigenvalues and eigenvectors

M|a> = λ_a |a>

- If M is a Hermitean matrx then λ_a is real
- Take inner product
  - <a|M|a> = λ_a |<a|a> (λ_a is a number)
  - <a|M|a> for M=hermitean → a is real
  - Therefore λ_a is real

- If M is an observable then the value λ_a is a measurable value.

44:00
The eigenvector (collection of λ eigenvalues) is the state of the system.

54:00 sigma matrices

sigma 3 z-axis (sometimes called the spin operator)

$$\sigma 3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

eigenvector {1 0},{0 -1}, eigenvalues +1,-1

$$\sigma 3|1\ \ 0> = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma 3|0\ -1> = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

sigma 1 x-axis

$$\sigma 1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigenvector {1 1},{1 -1}, eigenvalues +1,-1 (divided by √2)

$$\sigma 1|1\ \ 1> = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\sigma 1|1\ -1> = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

sigma 2 y-axis

$$\sigma 1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

eigenvector {1 i},{1 -i}, eigenvalues +1,-1

$$\sigma 2|1\ \ i> = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = +1\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\sigma 2|1\ -i> = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = -1\begin{pmatrix} -1 \\ i \end{pmatrix}$$

sigma identities

$\sigma 1^2 = \sigma 3^2 = \sigma 2^2 = I$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

σ1 σ2 = -i σ3 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

σ3 σ1 = -i σ2 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

σ2 σ3 = -i σ1 $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -i\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

note if you reverse the sigma matrix multiply order, the sign is reversed:
σ2 σ1 = +i σ3
σ1 σ3 = +i σ2
σ3 σ2 = +i σ1

60:00 observables, eigenvalues and eigenvectors are orthogonal

suppose an observable with 2 eigenvalue
(normally 3x3 M has 3, 4x4 has 4, …)

an observable with 2 eigenvalue
$M|a> = \lambda a\ |a>$ eigenvector a with eigenvalue λa
$M|b> = \lambda b\ |b>$ eigenvector b with eigenvalue λb

if λa & λb are different then they are orthogonal
(the vectors a,b are the states of the systems)

proof
1 $<b|M|a> = \lambda a <b|a>$ multiply by b
2 $<a|M|b> = \lambda b <a|b>$ multiply by a
take complex conjugate of 2
2a $<a|M|b>^* = \lambda b^* <a|b>^*$ or
2b $<b|M|a> = \lambda b\ <b|a>$
subtract 2b from 1 giving:
3 $0 = (\lambda a - \lambda b) <b|a>$
if a ne b, then a is orthogonal to b

--- exercise: check that σ vectors are orthogonal
use σ2 defn. and identities:
$\sigma 2 = \sigma 2^\dagger$; $\lambda a=+1, \lambda b=-1$
$|a\rangle = \begin{pmatrix}1\\ i\end{pmatrix}\ \langle a| = (1\quad -i)\ \ |b\rangle = \begin{pmatrix}1\\ -i\end{pmatrix}\ \langle b| = (1\quad i)$

4 $<b|\sigma 2|a> = \lambda a<b|a>$ $\quad (1\quad +i)\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\begin{pmatrix}1\\ +i\end{pmatrix} = +1(1\quad +i)\begin{pmatrix}1\\ +i\end{pmatrix}$

5 $<a|\sigma 2|b> = \lambda b<a|b>$ $\quad (1\quad -i)\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = -1(1\quad -i)\begin{pmatrix}1\\ -i\end{pmatrix}$

take complex conjugate of 5
5a $<a|\sigma 2|b>^* = \lambda b^*<a|b>^*$
5b $<a|\sigma 2|b>^* = (1\quad +i)\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}\begin{pmatrix}1\\ +i\end{pmatrix} = <b|\sigma 2|a>$
5c $\lambda b^*<a|b>^* = -1(1\quad +i)\begin{pmatrix}1\\ i\end{pmatrix} = \lambda b<b|a>$
5d $<b|\sigma 2|a> = \lambda b<b|a>$

subtracting 4-5d; as λa ≠ λb then $<b|a> = 0$
6 $0 = \lambda a<b|a> - \lambda b<b|a>$
notes:
1 as any $\sigma = \sigma^\dagger$, and a,b can be replaced with any σ eigenvector the above is sufficient for σ1, σ3
2 could have used (3) and just calculate $<b|a> = <a\}b> = 0$

72:00 can measure component of electron spin in any x,y,z
- but not simultaneously
- sigma vectors are the x, y, z components

let:
M be a system of observables; with λa, the probability of M being in state a (eigenvector a)

if you prepare the system as b – what is the probability that will be in the state a?
the probability is the square of the dot product of b and a

<a|b> <a|b>*

dot product of a,b times the dot product of the conjugate (square)

notes:
1: a,b are unit vectors;
2: <a|b> is a complex number, the square (complex conjugate) is a real number;
3: the probability, a real number, will always be ≤ 1;
4: if a,b are orthogonal the probability <a|b> is zero

80:00 if a,b were real vectors then the dot product squared <a|b><a|b> would be the cosine between them. But a,b are in general, complex so that the squre must be the complex conjugate.

82:00 prepare as σ3, test as σ1 – probability is ½

$|b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ prepared as σ3, which means σ3 = +1

this is done by placing a magnetic field in the “up” or “+z” direction. The observable, the spin, will be +1

now we will measure σ1 (horizontal “+x” direction) and ask what is the probability we will get +1?

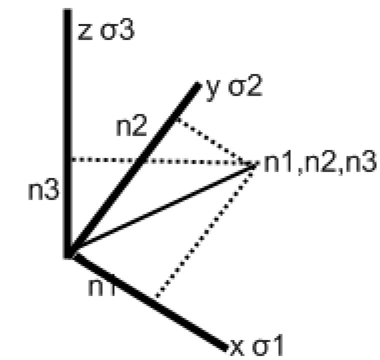
$|a\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ a is an eigenvector of σ1

the inner product <a|b> is (1/√2), the amplitude is (1/√2) (1/√2) = (1/2)

$|\langle a|b\rangle|^2 = \frac{1}{2}$ the form |<a|b>|^2 is convention for <a|b><a|b>*

similar probability for “-x”; (-1/√2) (-1/√2) = (1/2)
86:00 if line up spin in one direction, measure in another – probability is ½

93:00 example, unit pointer in any arbitrary direction.
--- always get +1 or -1 probabilities …



vector n (n1,n2,n3) projected onto sigma 3-D axis

n1 n2 n3 pointer components (unit normalized)

components of the spin are:

σ . n dot product

σ1*n1 + σ2*n2 + σ3*n3

multiplying each sigma matrix by n, then adding:

$$\begin{bmatrix} 0 & n1 \\ n1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -in2 \\ in2 & 0 \end{bmatrix} + \begin{bmatrix} n3 & 0 \\ 0 & -n3 \end{bmatrix} = \begin{bmatrix} n3 & n1 - in2 \\ n1 + in2 & -n3 \end{bmatrix}$$

we get a Hermitian matrix:

σ . n = σ1*n1 + σ2*n2 + σ3*n3 = $\begin{bmatrix} n3 & n1 - in2 \\ n1 + in2 & -n3 \end{bmatrix}$

let us square the dot product; (σ . n)*(σ . n) to show this equals one:

(σ1*n1 + σ2*n2 + σ3*n3) * (σ1*n1 + σ2*n2 + σ3*n3)

the normal terms equal one:

σ1*n1 * σ1*n1 + σ2*n2 * σ2*n2 + σ3*n3 * σ3*n3

grouping,

(n1*n1* σ1* σ1) + we know that any sigma squared = one, leaves us with:

n1*n1 + n2*n2 + n3*n3 + = 1 as vector n is unit normalized by defn.

the cross product terms all disappear:

σ1*n1 * σ2*n2 + σ1*n1 * σ3*n3 +
σ2*n2 * σ1*n1 + σ2*n2 * σ3*n3 +
σ3*n3 * σ1*n1 + σ3*n3 * σ2*n2

re-grouping:

σ1*n1 * σ2*n2 + σ2*n2 * σ1*n1 +
σ1*n1 * σ3*n3 + σ3*n3 * σ1*n1 +
σ2*n2 * σ3*n3 + σ3*n3 * σ2*n2

note that each line is equal to zero because of reverse order of matrix multiplication:

σ1*n1 * σ2*n2 = $\begin{bmatrix} 0 & n1 \\ n1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -in2 \\ in2 & 0 \end{bmatrix} = \begin{bmatrix} n1 + in2 & 0 \\ 0 & n1 - in2 \end{bmatrix}$

σ2*n2 * σ1*n1 = $\begin{bmatrix} 0 & -in2 \\ in2 & 0 \end{bmatrix} \begin{bmatrix} 0 & n1 \\ n1 & 0 \end{bmatrix} = \begin{bmatrix} n1 - in2 & 0 \\ 0 & n1 + in2 \end{bmatrix}$

or $(\sigma_1 * n_1 * \sigma_2 * n_2) = -(\sigma_2 * n_2 * \sigma_1 * n_1)$

the cross products equalling zero can be deduced by the sigma idenities above.