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4

General Formulation Methods

The formulation methods introduced in Chapter 2 are quite efficient and have been used successfully in many applications, but they cannot handle all ideal elements. To avoid the restrictions, general formulation methods are introduced in this chapter. In Section 4.1, the tableau formulation [1] is discussed. Here all branch currents, all branch voltages, and all nodal voltages are retained as unknown variables of the problem. Thus the formulation is most general (everything is available after the solution) but leads to large system matrices.

Section 4.2 indicates that blocks of variables can be eliminated and, under special circumstances, this naturally leads to the nodal formulation. However, if we wish to retain the ability to handle all types of network elements, complete block elimination is not possible and the modified nodal formulation [2] must be used. This can be done using graphs, as discussed in Section 4.3, or without graphs, as shown in Section 4.4.

The modified nodal formulations given in Sections 4.3 and 4.4 are efficient but still retain many redundant variables. It is demonstrated in Section 4.5 that active networks can be analyzed extremely efficiently if we follow a set of special rules. The rules given there cannot be easily used for computer solutions, and a systematic method must be found. The basis for eliminating redundant variables is the use of separate voltage and current graphs, discussed in Section 4.6, where they are applied to the tableau formulation. The graphs are a representation of the interconnections and, as such, can be replaced by tables which can be used for automated formulation. Such tabular representation is given in Section 4.7. With this background, the two-graph modified nodal formulation is developed in Section 4.8. Finally, Section 4.9 compares the various formulations introduced in this chapter, and Section 4.10 gives an example.

4.1. TABLEAU FORMULATION

The formulations discussed in the last two chapters can all be derived from a general formulation called the *tableau*. In this formulation, *all* equations describing the network are collected into one large matrix equation involving the KVL, the KCL, and the constitutive equations.

We will first comment on the most convenient type of tableau. For initial considerations, let the network have b branches; $n + 1$ nodes; R, G, L, and C elements; and sources. We can express the topological properties of such a network by means of the **A**, **Q**, and **B** matrices. The last two matrices are interdependent, and considerable effort is required to obtain them: a tree must be selected and the matrices brought into a proper form. It is much easier to work with the incidence matrix, and for this reason the tableau is based on it. Recall that the KCL was expressed by

$$\mathbf{A}\mathbf{I}_b = \mathbf{0} \tag{4.1.1}$$

whereas the KVL was given by

$$\mathbf{V}_b - \mathbf{A}^t\mathbf{V}_n = \mathbf{0}. \tag{4.1.2}$$

The subscript n stands for nodes, the subscript b for branches, and in applications b will be replaced by the element number. (For the fifth element, 5 will be written instead of b.)

The general representation describing all possible constitutive equations has the following form:

$$\begin{array}{l} \text{currents:} \\ \text{voltages:} \end{array} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{K}_2 \end{bmatrix} \mathbf{V}_b + \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{Z}_2 \end{bmatrix} \mathbf{I}_b = \begin{bmatrix} \mathbf{W}_{b_1} \\ \mathbf{W}_{b_2} \end{bmatrix}$$

where $\mathbf{Y}_1$ and $\mathbf{Z}_2$ represent admittances and impedances, respectively; $\mathbf{K}_1$ and $\mathbf{K}_2$ contain dimensionless constants; and $\mathbf{W}_{b_1}$ and $\mathbf{W}_{b_2}$ include the independent current and voltage sources, as well as the influence of initial conditions on capacitors and inductors. For notational compactness, we will use the following form:

$$\mathbf{Y}_b\mathbf{V}_b + \mathbf{Z}_b\mathbf{I}_b = \mathbf{W}_b. \tag{4.1.3}$$

In all subsequent formulations, capacitors will be entered in admittance form and inductors in impedance form to keep the variable s in the numerator. Since the Laplace transform variable s is equivalent to the differentiation operator, we will get a set of algebraic-differential equations when performing time domain analysis. Table 4.1.1 indicates the choices of Y_b, Z_b, and W_b for various two-terminal elements.

TABLE 4.1.1. Tableau Entries for Selected Elements

Element	Constitutive Equation	Value of Y_b	Value of Z_b	Value of W_b
Resistor	$V_b - R_bI_b = 0$	1	$-R_b$	0
Conductor	$G_bV_b - I_b = 0$	G_b	-1	0
Capacitor	$sC_bV_b - I_b = C_bV_0$	sC_b	-1	C_bV_0
Inductor	$V_b - sL_bI_b = -L_bI_0$	1	$-sL_b$	$-L_bI_0$
Voltage source	$V_b = E_b$	1	0	E_b
Current source	$I_b = J_b$	0	1	J_b

Equations (4.1.1) to (4.1.3) can be collected, for instance, in the following sequence:

$$\mathbf{V}_b - \mathbf{A}^t\mathbf{V}_n = \mathbf{0}$$

$$\mathbf{Y}_b\mathbf{V}_b + \mathbf{Z}_b\mathbf{I}_b = \mathbf{W}_b$$

$$\mathbf{A}\mathbf{I}_b = \mathbf{0}$$

and put into one matrix equation

$$\begin{array}{c|ccc} & b & b & n \\ \hline b & \mathbf{1} & \mathbf{0} & -\mathbf{A}^t \\ b & \mathbf{Y}_b & \mathbf{Z}_b & \mathbf{0} \\ n & \mathbf{0} & \mathbf{A} & \mathbf{0} \end{array} \begin{bmatrix} \mathbf{V}_b \\ \mathbf{I}_b \\ \mathbf{V}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{W}_b \\ \mathbf{0} \end{bmatrix} \tag{4.1.4}$$

or, in general,

$$\mathbf{T}\mathbf{X} = \mathbf{W}. \tag{4.1.5}$$

The arrangement indicated in (4.1.4) has square submatrices on the diagonal. In the tableau, there is no reason to distinguish between sources and passive elements, as we did in Chapter 3. The element numbering can be completely arbitrary. For theoretical considerations, one might wish to have special arrangements, depending on the purpose.

EXAMPLE 4.1.1. Write the tableau equations for the network and graph in Fig. 4.1.1.

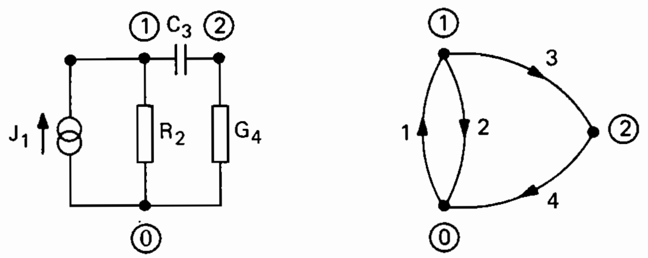


Fig. 4.1.1. Example demonstrating the tableau formulation.

The **A** matrix is

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \mathbf{A} = & \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \end{array}$$

In this example, both R and G elements were intentionally introduced. The tableau is

$$\left[\begin{array}{cccc:cccc:cc} 1 & & & & & & & & 1 & 0 \\ & 1 & & & & \mathbf{0} & & & -1 & 0 \\ & & 1 & & & & & & -1 & 1 \\ & & & 1 & & & & & 0 & -1 \\ \hdashline 0 & & & & 1 & & & & & \\ & 1 & & & & -R_2 & & & \mathbf{0} & \\ & & sC_3 & & & & -1 & & & \\ & & & G_4 & & & & -1 & & \\ \hdashline & \mathbf{0} & & & -1 & 1 & 1 & 0 & \mathbf{0} & \\ & & & & 0 & 0 & -1 & 1 & & \end{array}\right] \left[\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \hdashline I_1 \\ I_2 \\ I_3 \\ I_4 \\ \hdashline V_{n1} \\ V_{n2} \end{array}\right] = \left[\begin{array}{c} \\ \mathbf{0} \\ \\ \\ \hdashline J_1 \\ 0 \\ 0 \\ 0 \\ \hdashline \mathbf{0} \\ \\ \end{array}\right].$$

If the capacitor initially had a voltage V_0 across it, positive at node 1 and negative at node 2, the right-hand-side entry of the seventh row would be C_3V_0.

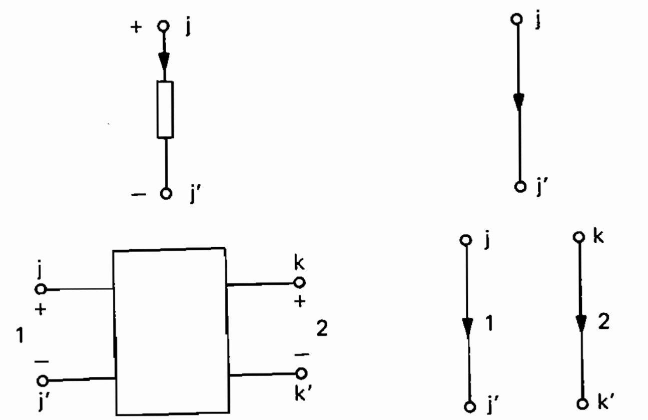


Fig. 4.1.2. One- and two-port networks and their graph representations.

Until now, we have been able to handle only two-terminal elements. We do not yet have a graph representation for the various ideal two-ports introduced in Chapter 1.

In order to generalize the tableau to any element, we will introduce first the simpler *one-graph concept* (to be distinguished from the two-graph concept introduced later). In the one-graph, each port of a two-port network is represented by an oriented line segment and two constitutive equations must be given. The graph representation is shown in Fig. 4.1.2. Two numbers are associated with each two-port when numbering the edges. The constitutive equations are precisely those discussed in Chapter 1 for the two-ports; the most important ideal elements are collected in Fig. 4.1.3.

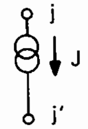
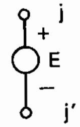
ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
CURRENT SOURCE		I = J
VOLTAGE SOURCE		V = E

Fig. 4.1.3. Constitutive equations of ideal elements for tableau formulation.

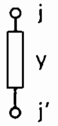
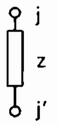
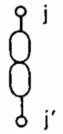
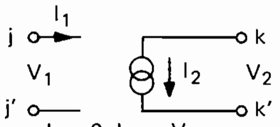
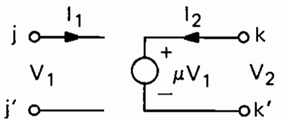
ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
OPEN CIRCUIT		$I = 0$
SHORT CIRCUIT		$V = 0$
ADMITTANCE		$yV - I = 0$
IMPEDANCE		$V - zI = 0$
NULLATOR		$I = 0$ $V = 0$
NORATOR		I, V ARBITRARY (NO CONSTITUTIVE EQUATIONS)
VCT	 $I_1 = 0;\ I_2 = gV_1$	$\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
VVT	 $I_1 = 0;\ V_2 = \mu V_1$	$\begin{bmatrix} 0 & 0 \\ \mu & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Fig. 4.1.3. (*Continued*)

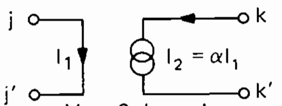
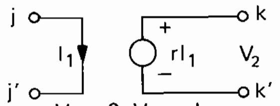
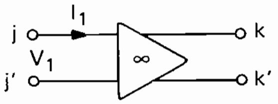
ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
CCT	 $V_1 = 0;\ I_2 = \alpha I_1$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
CVT	 $V_1 = 0;\ V_2 = rI_1$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
OPAMP	 $V_1 = 0;\ I_1 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Fig. 4.1.3. (*Continued*)

EXAMPLE 4.1.2. Write the **A** matrix and the constitutive matrices $\mathbf{Y}_b$, $\mathbf{Z}_b$ for the network and graph shown in Fig. 4.1.4.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Edge 6 denotes the input, edge 7 the output of the VVT, in agreement with Fig. 4.1.2. The constitutive equations for the elements are

$$V_1 = E_1$$

$$G_2V_2 - I_2 = 0$$

$$G_3V_3 - I_3 = 0$$

$$sC_4V_4 - I_4 = 0$$

$$sC_5V_5 - I_5 = 0$$

$$I_6 = 0$$

$$\mu V_6 - V_7 = 0.$$

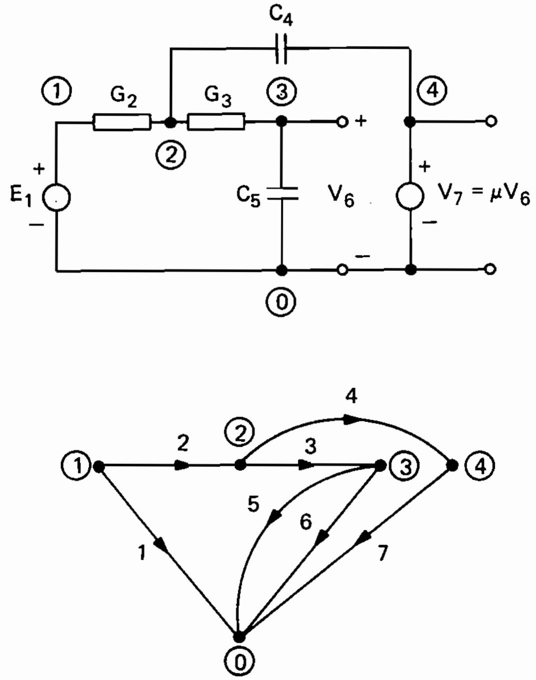


Fig. 4.1.4. Active network with a VVT and its graph.

The last two equations describe the VVT. The constitutive equations are rewritten in the form (4.1.3)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sC_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sC_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To compare various formulations, it is convenient to introduce the matrix density, defined as follows:

$$D = \frac{\text{number of nonzero entries in the matrix}}{\text{total number of all entries in the matrix}}. \tag{4.1.6}$$

For Example 4.1.2, the tableau has the size 18 × 18 and there are 39 nonzero entries. Thus the density becomes $D = 39/18^2 = 0.12$ or 12%.

EXAMPLE 4.1.3. The use of the OPAMP will be demonstrated on the generalized impedance convertor shown in Fig. 4.1.5. The two OPAMPs are rep-

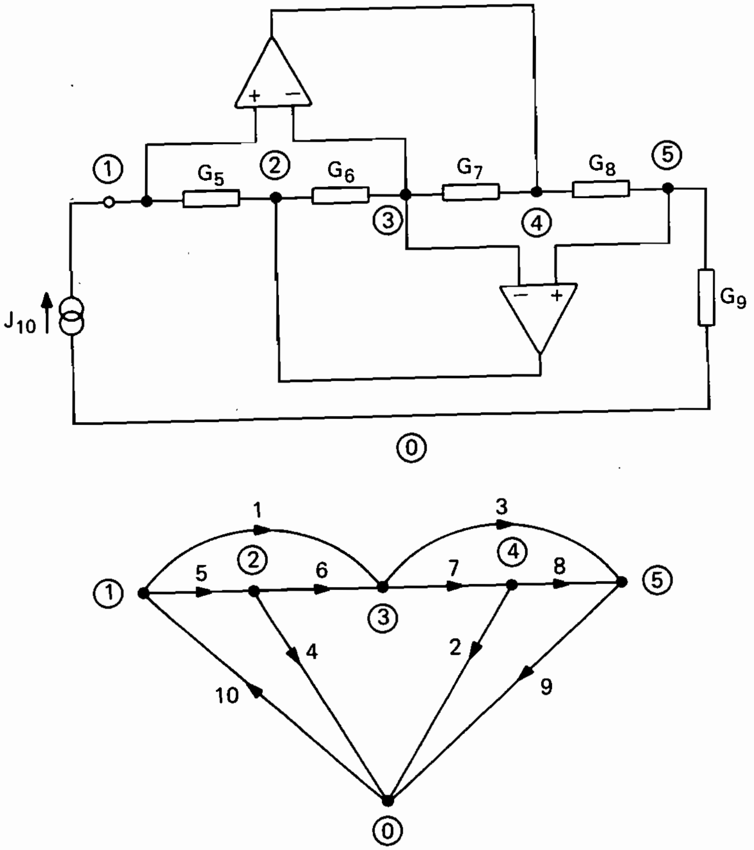


Fig. 4.1.5. Generalized impedance convertor and its graph.

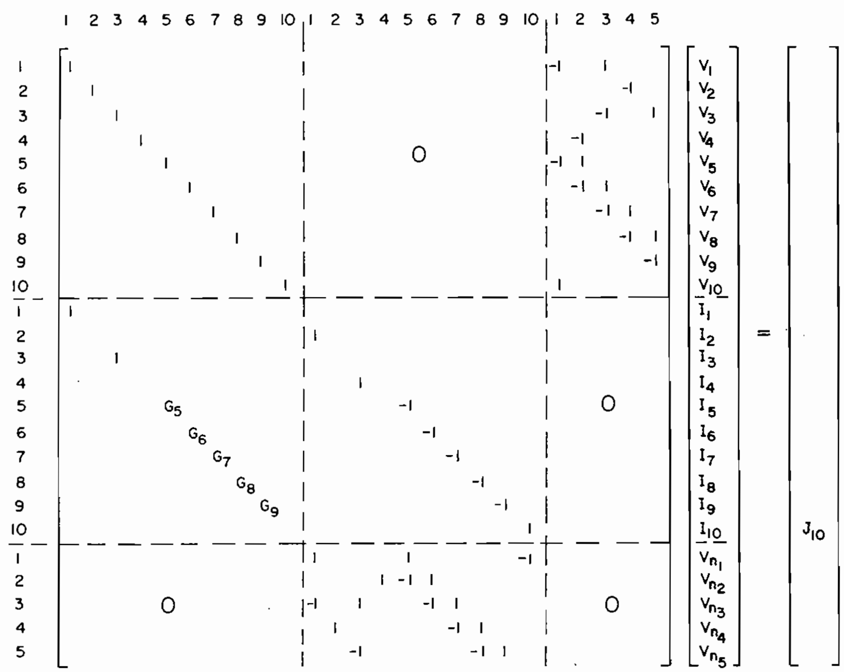


Fig. 4.1.6. One-graph tableau for the network in Fig. 4.1.5.

resented in the graph by the first four edges, the passive elements and the source by the remaining six. Write the tableau formulation for this network. The result is shown in Fig. 4.1.6. The matrix is of size 25 × 25, there are 57 nonzero entries, and the density is 9.12%.

The networks in Examples 4.1.2 and 4.1.3 will be used repeatedly in the rest of this chapter, and the sizes of the matrices will be given for various formulations. The tableau discussed in this section has mainly theoretical importance. The reader should note that many ideal two-ports introduce redundant variables: for instance, the input current of the VVT or VCT, or the input branch voltage of the CVT or CCT are known to be zero but they are kept in this formulation as variables. Elimination of such variables will be the subject of Section 4.6.

The tableau formulation has another problem: the resulting matrices are always quite large, and sparse matrix solvers are needed. Unfortunately, the structure of the matrix is such that coding these routines is complicated. Their treatment is beyond the scope of this book.

4.2. BLOCK ELIMINATION ON THE TABLEAU

In any network, the branch voltages are either equal to the node voltages (for grounded elements) or given by the difference of two node voltages (for elements connected between two nodes). Since this is a simple relationship and the node voltages are available after the tableau equations are solved, we can eliminate all branch voltages from the equations. Write the tableau equations again:

$$\mathbf{V}_b = \mathbf{A}^t\mathbf{V}_n$$

$$\mathbf{Y}_b\mathbf{V}_b + \mathbf{Z}_b\mathbf{I}_b = \mathbf{W}_b$$

$$\mathbf{A}\mathbf{I}_b = \mathbf{0}.$$

Substitute the first equation into the second:

$$\mathbf{Y}_b\mathbf{A}^t\mathbf{V}_n + \mathbf{Z}_b\mathbf{I}_b = \mathbf{W}_b \tag{4.2.1}$$

$$\mathbf{A}\mathbf{I}_b = \mathbf{0}. \tag{4.2.2}$$

In matrix form

$$\begin{bmatrix} \mathbf{Y}_b\mathbf{A}^t & \mathbf{Z}_b \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{I}_b \end{bmatrix} = \begin{bmatrix} \mathbf{W}_b \\ \mathbf{0} \end{bmatrix}. \tag{4.2.3}$$

The size of this matrix is $(b + n)$.

We can proceed even further under special circumstances. Assume that every element in the network is represented by its admittance. Then the branch current can be easily recovered either as $Y_bV_{n_j} = I_b$ or, for an ungrounded element, as $Y_b(V_{n_j} - V_{n_k}) = I_b$. In such cases, we can also eliminate the currents from (4.2.1) and (4.2.2). Let every element have the description

$$Y_bV_b - I_b = 0$$

and let only current sources be permitted:

$$I_b = J_b.$$

These two types of equations can be combined into a common matrix form:

$$\mathbf{I}_b = \mathbf{Y}_b\mathbf{V}_b + \mathbf{J}_b. \tag{4.2.4}$$

Substituting for $\mathbf{V}_b$ gives

$$\mathbf{I}_b = \mathbf{Y}_b\mathbf{A}^t\mathbf{V}_n + \mathbf{J}_b. \tag{4.2.5}$$

Inserting, finally, (4.2.5) into (4.2.2) gives

$$\mathbf{A}(\mathbf{Y}_b\mathbf{A}^t\mathbf{V}_n + \mathbf{J}_b) = 0$$

or

$$\mathbf{A}\mathbf{Y}_b\mathbf{A}^t\mathbf{V}_n = -\mathbf{A}\mathbf{J}_b. \tag{4.2.6}$$

This is exactly the nodal formulation discussed in (3.7.10) in Chapter 3. The size of the matrix is now only $n \times n$.

It must be noted that the admittance form does not exist for many useful ideal elements: voltage source, VVT, CVT, CCT, transformer, ideal transformer, OPAMP, norator, and nullator. Moreover, if we wish to preserve the variable s in the numerator, the inductor must be entered in its impedance form. In nonlinear networks (to be discussed later), a resistor may be a current-controlled device and its describing function may not be invertible. In all these cases, some currents must be retained in the formulation. We will present both formal and by-inspection methods for writing such formulations.

4.3. MODIFIED NODAL FORMULATION USING ONE GRAPH

This section presents the formal steps required in deriving the modified nodal formulation for all ideal elements. The idea underlying this formulation is to split the elements into groups; one group is formed by elements which have an admittance description and the other by those which do not. Then we can eliminate all branch currents for elements having the admittance description. This will partly fill the empty block in (4.2.3) and will reduce the number of unknown currents $\mathbf{I}_b$. Every element will be represented by the graphs given in Fig. 4.1.2. Initial conditions on inductors and capacitors will be replaced by equivalent sources (see Table 4.1.1).

Rearrange the elements of the network so that the KCL equation can be written in the following form:

$$[\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3]\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\\ \mathbf{J}\end{bmatrix} = \mathbf{0}. \tag{4.3.1}$$

The partitions are created so that:

1. $\mathbf{I}_1$ contains branch currents of elements that have an admittance representation and that are not required as solutions.
2. $\mathbf{I}_2$ contains branch currents for elements that do not have an admittance representation. It contains, in addition, branch currents of voltage sources *and* branch currents which are required as solutions.
3. $\mathbf{J}$ contains independent current sources.

The KVL equations are partitioned the same way:

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\\ \mathbf{V}_J\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1^t\\ \mathbf{A}_2^t\\ \mathbf{A}_3^t\end{bmatrix}\mathbf{V}_n. \tag{4.3.2}$$

Equation (4.3.2) in fact represents three separate matrix equations:

$$\mathbf{V}_1 = \mathbf{A}_1^t\mathbf{V}_n \tag{4.3.3}$$

$$\mathbf{V}_2 = \mathbf{A}_2^t\mathbf{V}_n \tag{4.3.4}$$

$$\mathbf{V}_J = \mathbf{A}_3^t\mathbf{V}_n. \tag{4.3.5}$$

Equation (4.3.5) is used to compute the voltages across the current sources once the $\mathbf{V}_n$ are found.

The branch relations for elements in partition 2 are

$$\mathbf{Y}_2\mathbf{V}_2 + \mathbf{Z}_2\mathbf{I}_2 = \mathbf{W}_2 \tag{4.3.6}$$

where the right-hand-side vector $\mathbf{W}_2$ contains nonzero entries only for the voltage sources.

The branch relations in the first partition are of the form

$$\mathbf{Y}_1\mathbf{V}_1 = \mathbf{I}_1. \tag{4.3.7}$$

Rewrite (4.3.1) in the following form:

$$\mathbf{A}_1\mathbf{I}_1 + \mathbf{A}_2\mathbf{I}_2 = -\mathbf{A}_3\mathbf{J}$$

and substitute (4.3.7) for $\mathbf{I}_1$:

$$\mathbf{A}_1\mathbf{Y}_1\mathbf{V}_1 + \mathbf{A}_2\mathbf{I}_2 = -\mathbf{A}_3\mathbf{J}.$$

Branch voltages $\mathbf{V}_1$ can be eliminated by substituting (4.3.3):

$$\mathbf{A}_1\mathbf{Y}_1\mathbf{A}_1^t\mathbf{V}_n + \mathbf{A}_2\mathbf{I}_2 = -\mathbf{A}_3\mathbf{J}. \tag{4.3.8}$$

Substituting similarly (4.3.4) into (4.3.6), obtain

$$\mathbf{Y}_2\mathbf{A}_2^t\mathbf{V}_n + \mathbf{Z}_2\mathbf{I}_2 = \mathbf{W}_2. \tag{4.3.9}$$

Equations (4.3.8) and (4.3.9) can be put into one matrix equation:

$$\begin{bmatrix} \mathbf{A}_1\mathbf{Y}_1\mathbf{A}_1^t & \mathbf{A}_2 \\ \mathbf{Y}_2\mathbf{A}_2^t & \mathbf{Z}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_3\mathbf{J} \\ \mathbf{W}_2 \end{bmatrix}. \tag{4.3.10}$$

Let us denote

$$\mathbf{A}_1\mathbf{Y}_1\mathbf{A}_1^t = \mathbf{Y}_{n_1} \tag{4.3.11}$$

$$-\mathbf{A}_3\mathbf{J} = \mathbf{J}_n. \tag{4.3.12}$$

Comparing this with the nodal formulation (4.2.6), we see that $\mathbf{Y}_{n_1}$ is the nodal admittance matrix for the elements in partition 1, while $\mathbf{J}_n$ represents the equivalent nodal current sources. Both $\mathbf{Y}_{n_1}$ and $\mathbf{J}_n$ can be written by inspection as explained in Chapter 2. The final form of the modified nodal formulation is

$$\begin{array}{r} \\ \text{KCL} \\ \\ \text{additional equations} \end{array} \begin{array}{c} \begin{array}{cc} \text{node voltages} & \text{additional currents} \end{array} \\ \begin{bmatrix} \mathbf{Y}_{n_1} & \mathbf{A}_2 \\ \mathbf{Y}_2\mathbf{A}_2^t & \mathbf{Z}_2 \end{bmatrix} \end{array} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_n \\ \mathbf{W}_2 \end{bmatrix} \begin{array}{l} \text{current sources applied to nodes} \\ \\ \text{influence of voltage sources.} \end{array} \tag{4.3.13}$$

Once (4.3.13) has been solved, the remaining currents are obtained from (4.3.7) and the branch voltages are obtained by using (4.3.2).

Note, for future reference, that the top equations express the KCL *at the nodes*. The system (4.3.13) retains the advantages of both the nodal and tableau methods.

EXAMPLE 4.3.1. Apply the modified nodal method to the network in Fig. 4.1.5.

The two OPAMPs are represented by the first four branches. The constitutive equations for the OPAMPs are taken from Fig. 4.1.3 and are put into one matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The submatrix $\mathbf{A}_2$ has the following form:

$$\mathbf{A}_2 = \begin{array}{c} \\ \text{nodes} \\ \downarrow \end{array} \begin{array}{c|cccc} & \multicolumn{4}{c}{\text{edges} \longrightarrow} \\ & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & -1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & -1 & 0 \end{array}$$

and the modified nodal equation is

$$\left[\begin{array}{ccccc:cccc} G_5 & -G_5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -G_5 & G_5 + G_6 & -G_6 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -G_6 & G_6 + G_7 & -G_7 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -G_7 & G_7 + G_8 & -G_8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -G_8 & G_8 + G_9 & 0 & 0 & -1 & 0 \\ \hdashline 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} V_{n_1} \\ V_{n_2} \\ V_{n_3} \\ V_{n_4} \\ V_{n_5} \\ \hdashline I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}\right] = \left[\begin{array}{c} J_{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

The size of the matrix is 9 × 9 (compared with the 25 × 25 tableau formulation). The number of nonzero entries is also reduced to 25. The density is increased to $D = 30.9\%$.

4.4. MODIFIED NODAL FORMULATION BY INSPECTION

The formal description of the modified nodal formulation by means of one graph eliminated all branch voltages plus branch currents for elements which have an admittance description. It cannot eliminate redundant variables which are known in advance ($I_1 = 0$ for a VVT, etc.). This elimination by means of graphs is possible if we introduce separate graphs for voltages and currents. However, based on the considerations and results obtained in Section 4.3, we can introduce a method which eliminates some of the redundant variables and, in addition, does not need any graphs or incidence matrices. The result is an important practical formulation method for computer application.

Using (4.3.11), enter by inspection all elements which have the admittance description. Denote the size of the **Y** part of the matrix by $m = n$, m being initially the number of ungrounded nodes. We will increase the size of the basic matrix whenever we enter an element which does not have an admittance description. Its constitutive equation will be attached at the bottom of the set of equations (as another row of the matrix), and the current flowing into the element will be attached to the system as a new variable (an additional column of the matrix).

To facilitate programming considerations, let us make some simple assumptions. Define two matrices **G** and **C** of equal sizes N (N larger than m). Define as well a vector **W** of length N for the right-hand-side vector of the system. The values of J, E, CV_0, and LI_0 will be entered into this vector. All conductors and frequency-independent numbers arising in the formulation will be stored in the matrix **G**, whereas capacitor and inductor values and other values that are associated with the frequency variable will be stored in the matrix **C**. All inductors will be considered in impedance form. Once the formulation part has been completed, and the matrices **G** and **C** and the vector **W** are prepared, the system matrix is obtained, for any s, by writing

$$\mathbf{T} = \mathbf{G} + s\mathbf{C}$$

and the solution may follow.

Assume that all conductances and VCTs are entered into **G** by means of the symbolic formulae discussed in Chapter 2. All capacitors are entered into the matrix **C**, and the current sources are placed properly into the source vector. So far, only the upper left corners of the matrices are occupied by network entries.

Consider now, for instance, a voltage source. It is given, along with all other elements, in Fig. 4.4.1. The *nodes* are denoted by j, j' and the voltage source enforces the condition

$$V_j - V_{j'} = E.$$

This constitutive equation is in terms of *nodal voltages* and is appended to the set of previously defined equations. In addition, a current I will flow between the terminals j, j'. To unify the notation, the currents at terminals j or k will be considered to be positive; thus for the voltage source $I_j = I$, $I_{j'} = -I$. They are taken into account in the KCL as a new variable: I in the jth row and $-I$ in the j'th row. The matrix and the source vector will have the following forms:

$$\begin{array}{c|cc:c|c} & V_j & V_{j'} & \overset{I}{m+1} & \text{right side} \\ \hline j & & & 1 & \\ j' & & & -1 & \\ \hdashline m+1 & 1 & -1 & & E \end{array}.$$

The increased size is indicated by the $(m + 1)$st row and column. Should one, say the jth, node of the voltage source be grounded, the jth row and column of the **G** and **C** matrices will not exist.

For a VVT (Fig. 4.4.1) the following equations apply:

$$\begin{aligned} -\mu V_j + \mu V_{j'} + V_k - V_{k'} &= 0, \\ I_j = I_{j'} &= 0 \\ I_k = -I_{k'} &= I. \end{aligned}$$

One row is added to the matrix, representing the constitutive equation. An additional column takes care of the unknown current I:

$$\begin{array}{c|cccc:c} & V_j & V_{j'} & V_k & V_{k'} & I \\ \hline j & & & & & \\ j' & & & & & \\ k & & & & & 1 \\ k' & & & & & -1 \\ \hdashline m+1 & -\mu & \mu & 1 & -1 & \end{array}.$$

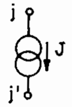
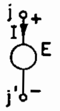
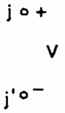
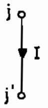
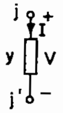
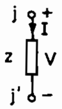
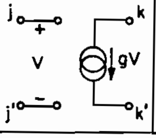
ELEMENT	SYMBOL	MATRIX	EQUATIONS
CURRENT SOURCE		$\begin{array}{c\|c\|} j & -J \\ j' & J \end{array}$ SOURCE VECTOR	$I_j = J$ $I_{j'} = -J$
VOLTAGE SOURCE		$\begin{array}{c\|ccc\|} & V_j & V_{j'} & I \\ \hline j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & \end{array}$ SOURCE VECTOR $\begin{bmatrix} \\ \\ E \end{bmatrix}$	$V_j - V_{j'} = E$ $I_j = I$ $I_{j'} = -I$
OPEN CIRCUIT		—	$V = V_j - V_{j'}$
SHORT CIRCUIT		$\begin{array}{c\|ccc\|} & V_j & V_{j'} & I \\ \hline j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & \end{array}$	$V_j - V_{j'} = 0$ $I_j = I$ $I_{j'} = -I$
ADMITTANCE		$\begin{array}{c\|cc\|} & V_j & V_{j'} \\ \hline j & y & -y \\ j' & -y & y \end{array}$	$I_j = y(V_j - V_{j'})$ $I_{j'} = -y(V_j - V_{j'})$
IMPEDANCE		$\begin{array}{c\|ccc\|} & V_j & V_{j'} & I \\ \hline j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & -z \end{array}$	$V_j - V_{j'} - zI = 0$ $I_j = -I_{j'} = I$
NULLATOR		$\begin{array}{c\|cc\|} & V_j & V_{j'} \\ \hline m+1 & 1 & -1 \end{array}$	$V_j - V_{j'} = 0$ $I_j = I_{j'} = 0$
NORATOR		$\begin{array}{c\|c\|} & I \\ \hline j & 1 \\ j' & -1 \end{array}$	V, I ARE ARBITRARY
VCT		$\begin{array}{c\|cc\|} & V_j & V_{j'} \\ \hline k & g & -g \\ k' & -g & g \end{array}$	$I_j = 0$ $I_{j'} = 0$ $I_k = g(V_j - V_{j'})$ $I_{k'} = -g(V_j - V_{j'})$

Fig. 4.4.1. Ideal elements in the modified nodal formulation without graphs.

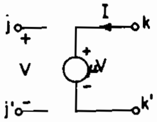
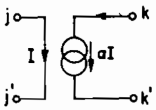
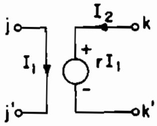
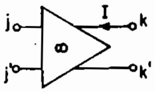
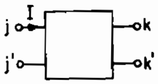
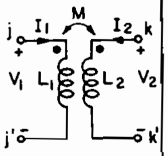
ELEMENT	SYMBOL	MATRIX	EQUATIONS
VVT		$\begin{array}{c\|ccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I \\ \hline j & & & & & \\ j' & & & & & \\ k & & & & & 1 \\ k' & & & & & -1 \\ m+1 & -\mu & \mu & 1 & -1 & \end{array}$	$-\mu V_j + \mu V_{j'} + V_k$ $-V_{k'} = 0$ $I_k = I$ $I_{k'} = -I$
CCT		$\begin{array}{c\|ccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I \\ \hline j & & & & & 1 \\ j' & & & & & -1 \\ k & & & & & a \\ k' & & & & & -a \\ m+1 & 1 & -1 & & & \end{array}$	$V_j - V_{j'} = 0$ $I_j = -I_{j'} = I$ $I_k = -I_{k'} = aI$
CVT		$\begin{array}{c\|cccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \\ \hline j & & & & & 1 & \\ j' & & & & & -1 & \\ k & & & & & & 1 \\ k' & & & & & & -1 \\ m+1 & 1 & -1 & & & & \\ m+2 & & & 1 & -1 & -r & \end{array}$	$V_j - V_{j'} = 0$ $V_k - V_{k'} - rI_1 = 0$ $I_j = -I_{j'} = I_1$ $I_k = -I_{k'} = I_2$
OPERATIONAL AMPLIFIER		$\begin{array}{c\|ccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I \\ \hline j & & & & & \\ j' & & & & & \\ k & & & & & 1 \\ k' & & & & & -1 \\ m+1 & 1 & -1 & & & \end{array}$	$V_j - V_{j'} = 0$ $I_k = -I_{k'} = I$
CONVERTOR		$\begin{array}{c\|ccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I \\ \hline j & & & & & 1 \\ j' & & & & & -1 \\ k & & & & & -K_2 \\ k' & & & & & K_2 \\ m+1 & 1 & -1 & -K_1 & K_1 & \end{array}$	$V_j - V_{j'} - K_1V_k + K_1V_{k'} = 0$ $I_j = -I_{j'} = I$ $I_k = -I_{k'} = -K_2I$ FOR IDEAL TRANSFORMER $K_1 = K_2 = n$
TRANSFORMER		$\begin{array}{c\|cccccc\|} & V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \\ \hline j & & & & & 1 & \\ j' & & & & & -1 & \\ k & & & & & & 1 \\ k' & & & & & & -1 \\ m+1 & 1 & -1 & & & -sL_1 & -sM \\ m+2 & & & 1 & -1 & -sM & -sL_2 \end{array}$	$V_j - V_{j'} - sL_1I_1 - sMI_2 = 0$ $V_k - V_{k'} - sMI_1 - sL_2I_2 = 0$ $I_j = -I_{j'} = I_1$ $I_k = -I_{k'} = I_2$

Fig. 4.4.1. (*Continued*)

The reader should have no difficulty in deriving the formulae for the remaining ideal elements. They are all collected in Fig. 4.4.1.

It is interesting to show that even a perfect switch can be incorporated into the formulation. Consider a conductance for which we wish to retain the current as the variable available upon the solution of the system. The constitutive equation for a conductance between nodes j, j' is

$$G(V_j - V_{j'}) - I = 0$$

and this equation is appended to the system of equations. The current is taken into account by an additional column:

$$\begin{array}{c|cc|c|} & j & j' & m+1 \\ j & & & 1 \\ j' & & & -1 \\ \cline{2-4} m+1 & G & -G & -1 \end{array} .$$

For a resistor, the entries are in Fig. 4.4.1 and are repeated here:

$$\begin{array}{c|cc|c|} & j & j' & m+1 \\ j & & & 1 \\ j' & & & -1 \\ \cline{2-4} m+1 & 1 & -1 & -R \end{array} .$$

An open circuit requires $G = 0$, while $R = 0$ results in a short circuit. We can thus combine the above representations as follows:

$$\begin{array}{c|cc|c|} & j & j' & m+1 \\ j & & & 1 \\ j' & & & -1 \\ \cline{2-4} m+1 & F & -F & F-1 \end{array} \qquad (4.4.1)$$

and select the value of F according to the following scheme:

Condition	F
Open circuit	0
Short-circuit	1

If the system matrix is generated by means of (4.4.1), the switches can be opened or closed without reformulating the equations. Only the proper value for F is inserted in the matrix before the solution.

We next present two examples to demonstrate the writing of the modified nodal equations.

EXAMPLE 4.4.1. Write the modified nodal formulation for the network of Fig. 4.1.4 by inspection.

Using the node numbering in the figure, write the nodal admittance portion for the conductors and capacitors. Then append the equation for the voltage source and finally for the VVT. Denote $I_1 = I_E$, $I_7 = I_{\mathrm{VVT}}$.

$$\left[\begin{array}{cccc|cc} G_2 & -G_2 & 0 & 0 & 1 & 0 \\ -G_2 & G_2 + G_3 + sC_4 & -G_3 & -sC_4 & 0 & 0 \\ 0 & -G_3 & G_3 + sC_5 & 0 & 0 & 0 \\ 0 & -sC_4 & 0 & sC_4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ \hline I_E \\ I_{\mathrm{VVT}} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline E_1 \\ 0 \end{array}\right] .$$

EXAMPLE 4.4.2. Write the modified nodal formulation for the network of Fig. 4.1.5.

$$\left[\begin{array}{ccccc|cc} G_5 & -G_5 & 0 & 0 & 0 & 0 & 0 \\ -G_5 & G_5 + G_6 & -G_6 & 0 & 0 & 0 & 1 \\ 0 & -G_6 & G_6 + G_7 & -G_7 & 0 & 0 & 0 \\ 0 & 0 & -G_7 & G_7 + G_8 & -G_8 & 1 & 0 \\ 0 & 0 & 0 & -G_8 & G_8 + G_9 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} V_{n1} \\ V_{n2} \\ V_{n3} \\ V_{n4} \\ V_{n5} \\ \hline I_{\mathrm{OP1}} \\ I_{\mathrm{OP2}} \end{array}\right] = \left[\begin{array}{c} J_{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] .$$

For the network of Fig. 4.1.4, the original 18 × 18 tableau matrix was reduced to 6 × 6. For the example of Fig. 4.1.5, the original 25 × 25 tableau matrix was reduced to 7 × 7 without losing any relevant information. We still calculate the current of the voltage source, the current of the source-port of the VVT, and the current of the output port of the OPAMPs.

4.5. NODAL ANALYSIS OF ACTIVE NETWORKS

Active networks are, in most cases, realized by means of VVTs or OPAMPs. They are usually excited by a voltage source, and their output is usually a voltage. A direct application of the nodal admittance concept is not possible, but considerable reduction of the system matrix size is possible if we apply some simple preprocessing steps.

The method presented here is intended for hand calculations but is also an introduction to the formal methods presented in the following sections. We will start with the assumption that *one terminal of each voltage source*, dependent or independent, is grounded.

If the voltage source has one grounded terminal, the other terminal voltage is known. The output voltage of a VVT depends on some voltage elsewhere, but once that has been specified, the voltage of the source port is known as well. If we are not interested in the current flowing through the source, we do not have to write the KCL for the node to which it is connected.

The above facts can be combined into the following rules for writing the equations of a network with grounded voltage sources:

1. Insert the known voltages into the circuit diagram. Each node must have a voltage, known or unknown.
2. Denote all resistors by their conductances: $G_i = 1/R_i$.
3. Write the KCL equations for the nodes *not* connected to independent or dependent voltage sources.

To illustrate the procedure, the rules will be applied to several examples. We also introduce the usual active network symbol for the VVT, shown in Fig. 4.5.1.

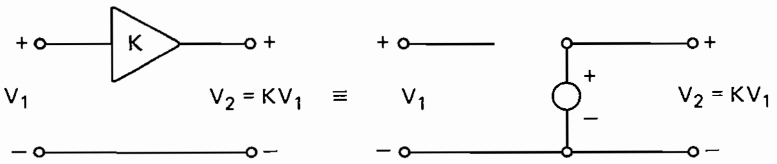


Fig. 4.5.1. Symbol for an amplifier and its VVT equivalent.

EXAMPLE 4.5.1. Write the network equations for the network shown in Fig. 4.5.2.

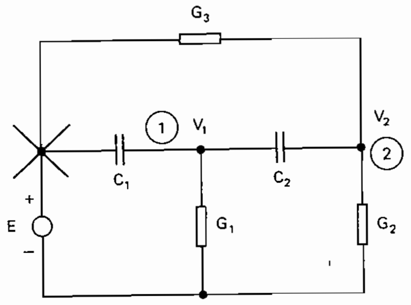


Fig. 4.5.2. Network with an independent voltage source.

The voltages are indicated in the figure, and the node with the voltage source is marked with a cross to indicate that the KCL equation is not written there. Every node must be assigned a voltage, as shown in the figure. The KCL equations are written for nodes 1 and 2 only:

$$(G_1 + sC_1 + sC_2)V_1 - sC_2V_2 = sC_1E$$

$$-sC_2V_1 + (G_2 + G_3 + sC_2)V_2 = G_3E.$$

It is convenient to rewrite them as a matrix equation:

$$\begin{bmatrix} G_1 + sC_1 + sC_2 & -sC_2 \\ -sC_2 & G_2 + G_3 + sC_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} sC_1E \\ G_3E \end{bmatrix}.$$

EXAMPLE 4.5.2. Apply the method to the network in Fig. 4.1.4.

Nodes 1 and 4 are not considered for the KCL, and equations are written only at nodes 2 and 3. We also know that $V_4 = \mu V_3$.

KCL at node 2:

$$(G_2 + G_3 + sC_4)V_2 - G_3V_3 - sC_4\mu V_3 = G_2E_1.$$

KCL at node 3:

$$-G_3V_2 + (G_3 + sC_5)V_3 = 0.$$

In matrix form, this is

$$\begin{bmatrix} G_2 + G_3 + sC_4 & -(G_3 + sC_4\mu) \\ -G_3 & G_3 + sC_5 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} G_2E_1 \\ 0 \end{bmatrix}.$$

EXAMPLE 4.5.3. Apply the method to the network in Figure 4.5.3.

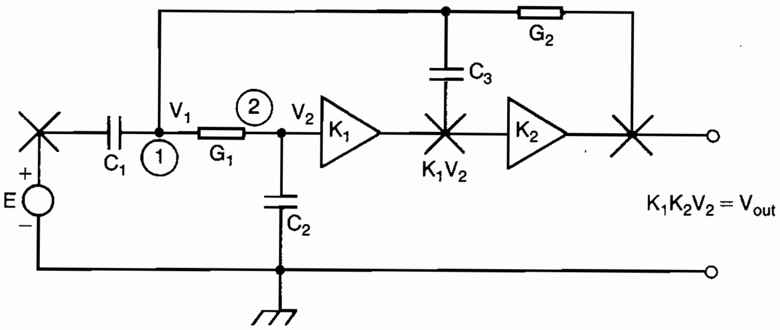


Fig. 4.5.3. Network with two amplifiers.

The nodes where we do not write the KCL are crossed out; equations are written only for nodes 1 and 2:

$$(G_1 + G_2 + sC_1 + sC_3)V_1 - G_1V_2 - sC_3K_1V_2 - G_2K_1K_2V_2 = sC_1E$$

$$-G_1V_1 + (G_1 + sC_2)V_2 = 0$$

In matrix form:

$$\begin{bmatrix} (G_1 + G_2 + sC_1 + sC_3) & -(G_1 + sC_3K_1 + G_2K_1K_2) \\ -G_1 & (G_1 + sC_2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} sC_1E \\ 0 \end{bmatrix}.$$

Analysis of networks with ideal OPAMPs is based on the same idea. The ideal OPAMP was introduced in Fig. 1.6.5. Its output is a voltage source, ideally with infinitely large gain, acting on the difference of the two input voltages. Marking the output with subscript o, we can write

$$(V_+ - V_-)A = V_o \tag{4.5.1}$$

It is convenient to introduce the *inverted* gain at this point

$$B = -\frac{1}{A}. \tag{4.5.2}$$

The reason for the minus sign will become clear in Section 5.3. The advantage of the inversion is that infinity cannot be handled by computers but zero can. Equation (4.5.1) changes into

$$V_+ - V_- + BV_o = 0. \tag{4.5.3}$$

If the OPAMP gain approaches infinity, $A \to \infty$, then B is simply set equal to zero and

$$V_+ = V_-.$$

We conclude that the voltages at the input terminals of an ideal OPAMP must be equal. The rules stated above are supplemented by:

4. Write equal voltages at the input terminals of the ideal OPAMP. If one of the terminals is grounded, the other one will also be at zero potential. Do not write the KCL equation at the output node of the OPAMP.

EXAMPLE 4.5.4. Analyze the network in Fig. 4.5.4.

Node 1 is at the same potential, E, as the other input node. The KCL is written at node 1 only:

$$(G_1 + G_2)E - G_2V_{\text{out}} = 0.$$

The network acts as a VVT, with gain defined by the ratio of the conductances.

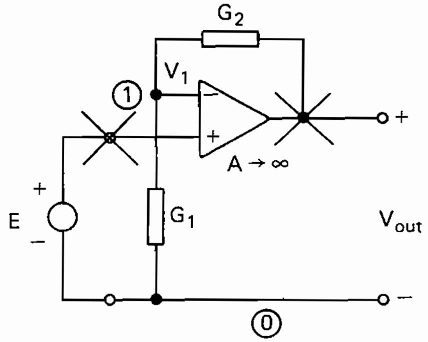


Fig. 4.5.4. OPAMP realization of a VVT with positive gain.

EXAMPLE 4.5.5. Apply the above rules to write the equations for the network in Fig. 4.5.5.

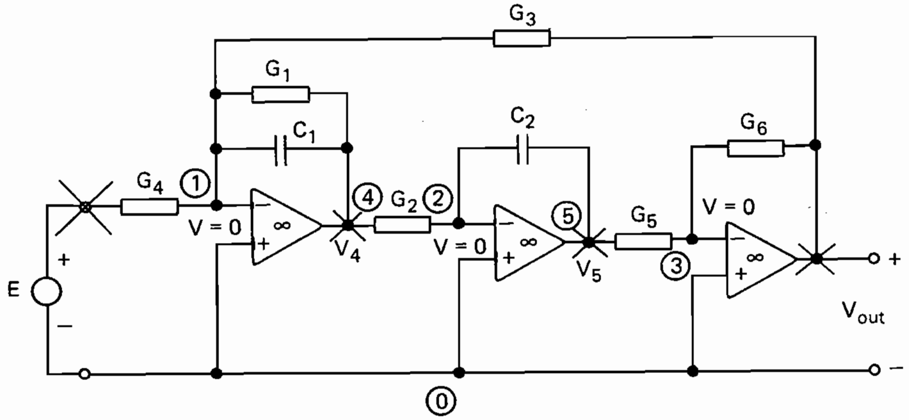


Fig. 4.5.5. Active network with three OPAMPs.

One input terminal of each OPAMP is grounded; the voltage at the other input node must be zero. Denote the nonzero voltages by V_4, V_5, V_{out}. The nodes with voltage sources are marked by a cross to indicate that no equations are written there. We write equations only at the nodes denoted as 1, 2, and 3:

$$0(G_1 + G_2 + G_4 + sC_1) - V_4(G_1 + sC_1) - V_{\text{out}}G_3 = G_4E$$

$$0(G_2 + sC_2) - V_4G_2 - V_5sC_2 = 0$$

$$0(G_5 + G_6) - V_5G_5 - V_{\text{out}}G_6 = 0$$

Here we have intentionally retained the terms multiplied by zero voltage for better understanding, but there was no need to write them. Putting the remaining terms into a matrix equation, we get

$$\begin{bmatrix} -G_1 - sC_1 & 0 & -G_3 \\ -G_2 & -sC_2 & 0 \\ 0 & -G_5 & -G_6 \end{bmatrix} \begin{bmatrix} V_4 \\ V_5 \\ V_{\text{out}} \end{bmatrix} = \begin{bmatrix} G_4E \\ 0 \\ 0 \end{bmatrix}.$$

EXAMPLE 4.5.6. Write the necessary equations for the network in Fig. 4.5.6. The OPAMPs are ideal.

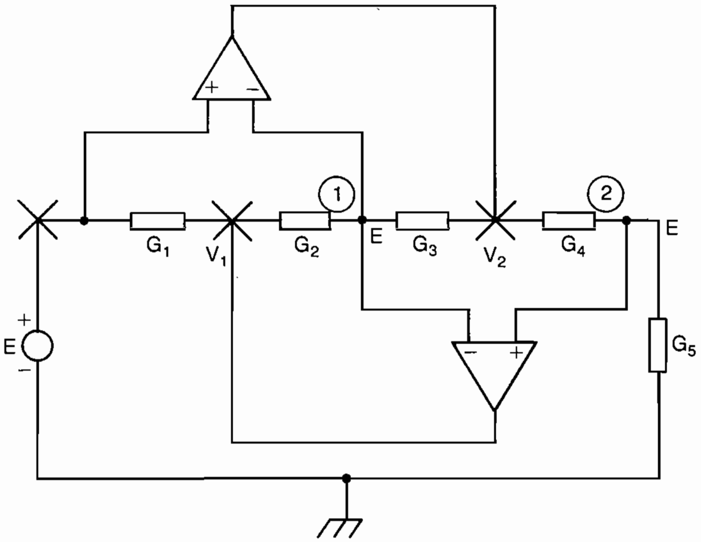


Fig. 4.5.6. Network with two ideal OPAMPs.

Notice that ideal OPAMPs force three node voltages of the network to be equal to the input voltage, E. The nodes with voltage sources are marked with crosses, and only two equations need be written:

$$E(G_2 + G_3) - V_1G_2 - V_2G_3 = 0$$

$$E(G_4 + G_5) - G_4V_2 = 0.$$

Note that we have written the KCL at the nodes with numbers enclosed in circles, using the voltage symbols shown on the figure. In matrix form:

$$\begin{bmatrix} G_2 & G_3 \\ 0 & G_4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} E(G_2 + G_3) \\ E(G_4 + G_5) \end{bmatrix}.$$

If the OPAMPs are nonideal, we can no longer assume that the input terminals are at the same potential, but the method still remains valid. All nodes are assigned different voltages, equations are not written at the nodes marked with crosses, and use is made of (4.5.2) and (4.5.3). We will apply the method to two problems.

EXAMPLE 4.5.7. Write the equations for the network in Fig. 4.5.7. The OPAMPs are nonideal.

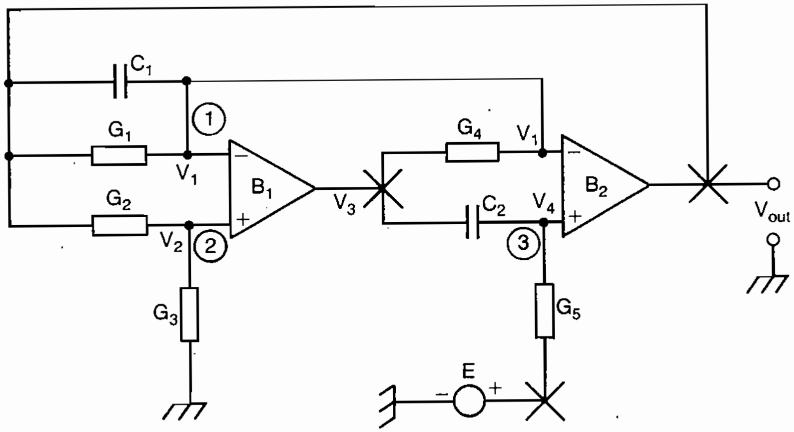


Fig. 4.5.7. Network with two nonideal OPAMPs.

All the preliminary operations have been written into the figure. We give only the matrix form:

$$\begin{bmatrix} (G_1+G_4+sC_1) & 0 & -G_4 & 0 & -(G_1+sC_1) \\ 0 & (G_2+G_3) & 0 & 0 & -G_2 \\ 0 & 0 & -sC_2 & (G_5+sC_2) & 0 \\ -1 & 1 & B_1 & 0 & 0 \\ -1 & 0 & 0 & 1 & B_2 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_{\text{out}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ EG_5 \\ 0 \\ 0 \end{bmatrix}.$$

EXAMPLE 4.5.8. Write the nodal equations for the network in Fig. 4.5.8. The OPAMPs are nonideal.

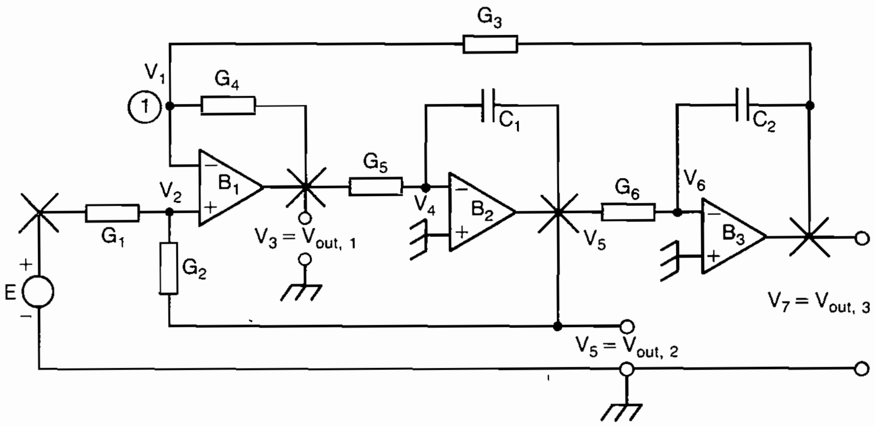


Fig. 4.5.8. Network with three nonideal OPAMPs.

We write four nodal equations:

$$(G_3+G_4)V_1 - G_4V_3 - G_3V_7 = 0$$

$$(G_1+G_2)V_2 - G_2V_5 = EG_1$$

$$-G_5V_3 + (G_5+sC_1)V_4 - sC_1V_5 = 0$$

$$-G_6V_5 + (G_6+sC_2)V_6 - sC_2V_7 = 0$$

followed by the equations describing the properties of the operational amplifiers:

$$-V_1 + V_2 + B_1V_3 = 0$$

$$-V_4 + B_2V_5 = 0$$

$$-V_6 + B_3V_7 = 0.$$

The matrix equation is

$$\begin{bmatrix} (G_3+G_4) & 0 & -G_4 & 0 & 0 & 0 & -G_3 \\ 0 & (G_1+G_2) & 0 & 0 & -G_2 & 0 & 0 \\ 0 & 0 & -G_5 & (G_5+sC_1) & -sC_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -G_6 & (G_6+sC_2) & -sC_2 \\ -1 & +1 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & B_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & B_3 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} 0 \\ EG_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To summarize the development thus far, we were able to eliminate all currents of the voltage sources from the equations; the only variables are nodal voltages. So far, all voltage sources had one of their terminals grounded.

The situation changes considerably if the voltage sources are floating, but it is still possible to eliminate the currents. In order to derive additional rules, first consider the network with one floating independent voltage source, Fig. 4.5.9.

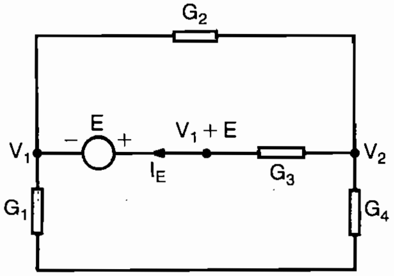


Fig. 4.5.9. Network with a floating independent voltage source.

One terminal of the voltage source is assigned the voltage V_1. The other has the voltage $V_1 + E$. If we write the KCL at all nodes, then we get the system

$$V_1(G_1+G_2) - G_2V_2 - I_E = 0$$

$$(V_1+E)G_3 - V_2G_3 + I_E = 0$$

$$-G_2V_1 - (V_1+E)G_3 + (G_2+G_3+G_4)V_2 = 0.$$

We are not interested in the current I_E, and its elimination can be accomplished by *adding* the first two equations:

$$(G_1+G_2+G_3)V_1 - (G_2+G_3)V_2 = -EG_3$$

$$-(G_2+G_3)V_1 + (G_2+G_3+G_4)V_2 = EG_3.$$

In matrix form:

$$\begin{bmatrix} G_1+G_2+G_3 & -(G_2+G_3) \\ -(G_2+G_3) & G_2+G_3+G_4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -EG_3 \\ EG_3 \end{bmatrix}.$$

Let us now discuss how we can take advantage of the above steps. We can consider both ends of the voltage source as one node when writing the KCL, but we must preserve the nodal voltages as they actually are. We could say that we are *collapsing* the two nodes into one when writing the KCL.

EXAMPLE 4.5.9. Write the nodal equations for the network Fig. 4.5.10, using the concept of collapsing the nodes when writing the KCL. The OPAMP is ideal.

The input nodes of the OPAMP are at the same potential, as indicated. The independent voltage source is grounded, and we can mark the node with a cross. We will still write the KCL with the current I_{op}, to show that the rule of adding the KCL of the output terminals is valid.

$$(G_1+G_2+G_7)V_1 - G_7V_2 = EG_1$$

$$(G_4+G_5+G_6)V_1 - G_6V_3 = EG_4$$

$$-G_7V_1 + (G_3+G_7)V_2 + I_{op} = EG_3$$

$$-G_6V_1 + G_6V_3 - I_{op} = 0.$$

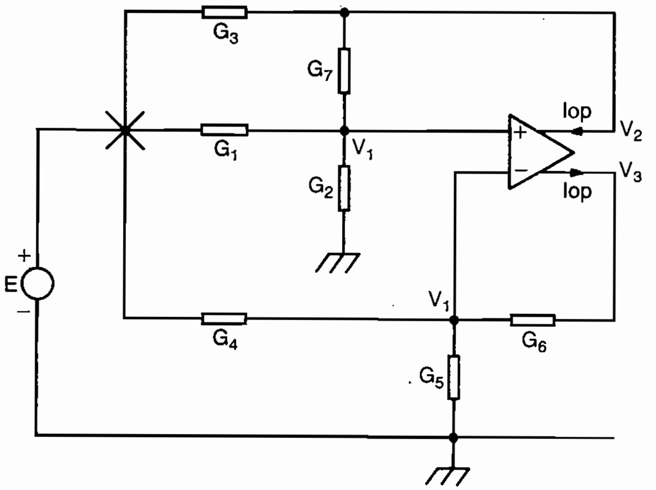


Fig. 4.5.10. Network with a floating OPAMP.

Elimination of the current I_{op} is achieved by adding the last two equations; this is equivalent to collapsing the nodes when writing the KCL. This results in the following system matrix:

$$\begin{bmatrix} G_1 + G_2 + G_7 & -G_7 & 0 \\ G_4 + G_5 + G_6 & 0 & -G_6 \\ -(G_6 + G_7) & G_3 + G_7 & G_6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} EG_1 \\ EG_4 \\ EG_3 \end{bmatrix}.$$

If the amplifier does not have infinite gain, the voltages at its input terminals cannot be considered equal. We will use the same network as in the previous example, but let the amplifier gain be A.

EXAMPLE 4.5.10. Write the equations for the network in Fig. 4.5.11. Collapse the output nodes of the amplifier without considering its currents.

The equations are

$$\begin{aligned} (G_1 + G_2 + G_7)V_1 - G_7[V_3 + A(V_1 - V_2)] &= EG_1 \\ (G_4 + G_5 + G_6)V_2 - G_6V_3 &= EG_4 \\ -G_7V_1 + [V_3 + A(V_1 - V_2)](G_3 + G_7) - G_6V_2 + G_6V_3 &= EG_3. \end{aligned}$$

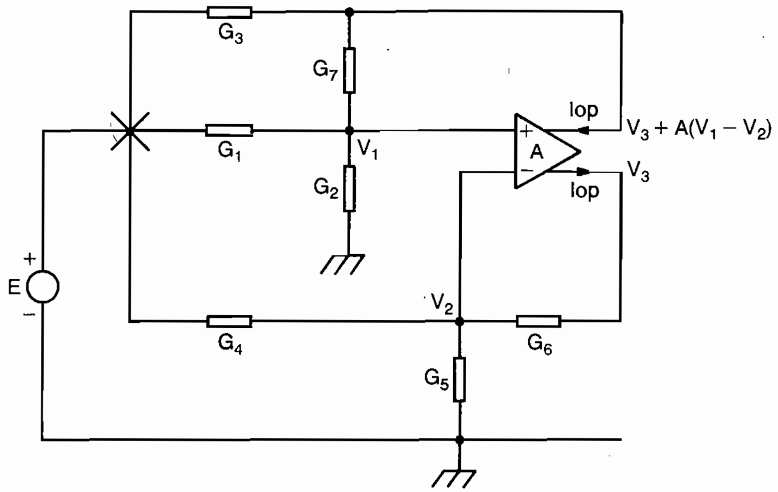


Fig. 4.5.11. Network with a floating amplifier.

Rearranging terms, this results in

$$\begin{bmatrix} G_1 + G_2 + G_7(1 - A) & G_7A & -G_7 \\ 0 & G_4 + G_5 + G_6 & -G_6 \\ -G_7(1 - A) + AG_3 & -G_6 - AG_3 - AG_7 & G_3 + G_6 + G_7 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} EG_1 \\ EG_4 \\ EG_3 \end{bmatrix}.$$

The above developments will help understand the two-graph method, the subject of the next section.

4.6. SEPARATE CURRENT AND VOLTAGE GRAPHS

For two-terminal elements, an edge on a single graph simultaneously represents the current through it and the voltage across it. For two-port networks, one of the variables may be zero, for instance, the input port current of a VVT. In order to handle two-ports by a single graph, we added the input current to the constitutive equations and allowed separate edges for the input and the output.

Separate voltage and current graphs offer a way to eliminate this redundancy. If the current into the port is zero, the edge is omitted on the *I*-graph but is kept on the *V*-graph.

A study of the various possibilities leads to the following set of rules for drawing the *I*- and *V*-graphs:

1. If the current in the branch *does not enter the constitutive equations* and is of no interest, its edge is collapsed on the *I*-graph.
2. If the current in the branch is zero, its edge is deleted from the *I*-graph.
3. If the voltage across the branch *does not enter the constitutive equations* and is of no interest, the edge is deleted on the *V*-graph.
4. If the voltage across the branch is zero, its edge is collapsed on the *V*-graph.

Note that the edge of a variable which enters the constitutive equations cannot be collapsed. The words "of no interest" imply that the particular variable will not be needed as the solution of the system; otherwise, the edge must be retained on the graph. For instance, one is often not interested in the current through a voltage source or the voltage across a current source.

If the rules are applied, the graphs may not only differ in structure but may even have a different number of nodes and edges. The incidence matrix of the *I*-graph is used to write the KCL, whereas the incidence matrix of the *V*-graph is used for the KVL:

$$\begin{aligned} \mathbf{V}_b &= \mathbf{A}_v^t \mathbf{V}_n \\ \mathbf{Y}_b \mathbf{V}_b + \mathbf{Z}_b \mathbf{I}_b &= \mathbf{W}_b \\ \mathbf{A}_i \mathbf{I}_b &= \mathbf{0}. \end{aligned} \tag{4.6.1}$$

The subscripts i, v refer to the I or V-graph; $\mathbf{Y}_b$ and $\mathbf{Z}_b$ need no longer be square matrices.

The rules stated above were applied to all ideal elements and are collected in Fig. 4.6.1. It was assumed that the voltage across the current source (dependent or independent) or the current through the voltage source (dependent or independent) is of no interest. All four transducers now have only one constitutive equation; the variable whose value is known to be zero is eliminated. Two-ports which need two equations for their full description (gyrator, converter, inverter, and transformer) are represented as in the one-graph method.

Since collapsing of nodes requires renumbering, we will use the following notation:

Original nodes of the network will be denoted by numbers in circles.
Renumbered nodes of the *I*-graph will be denoted by numbers in squares.
Renumbered nodes of the *V*-graph will be denoted by numbers in triangles.

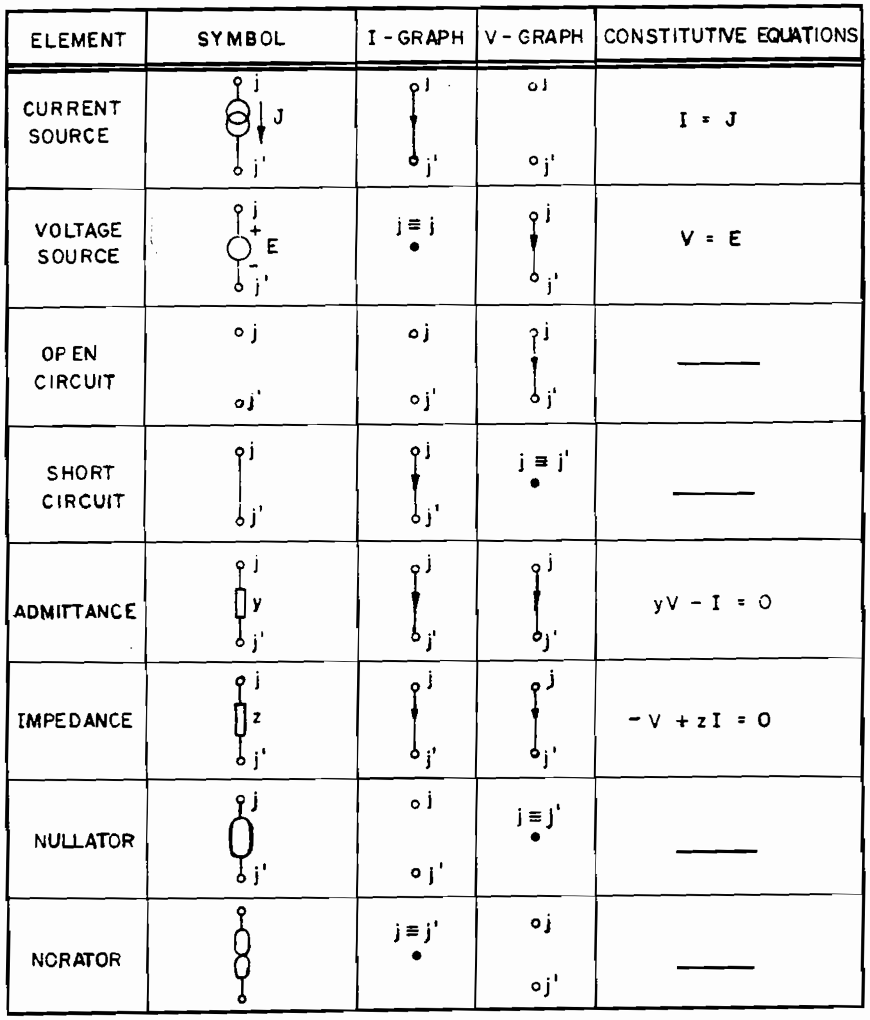


Fig. 4.6.1. Ideal elements and their two-graph representation.

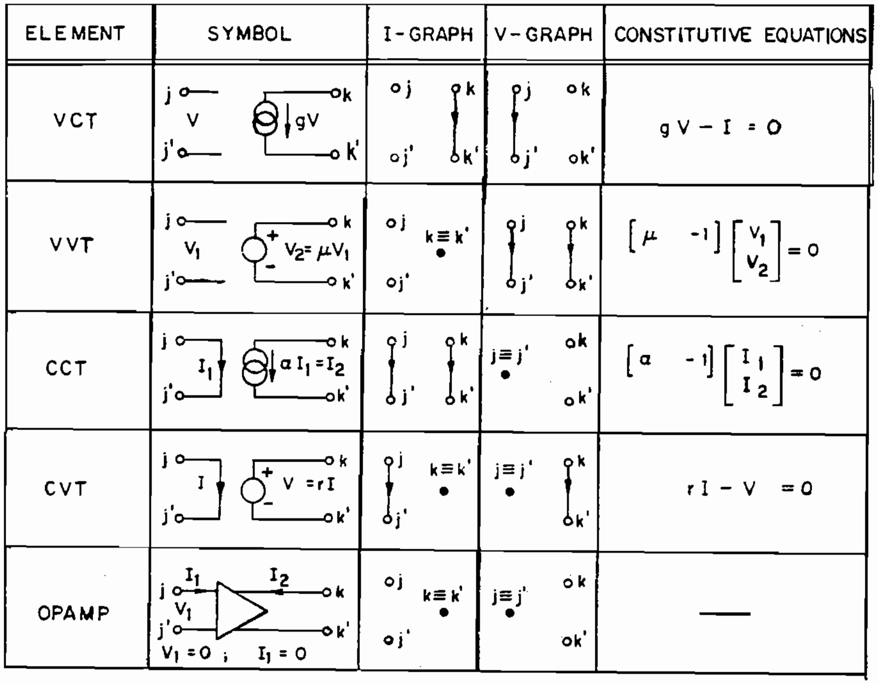


Fig. 4.6.1. (*Continued*)

The tableau matrix equation is easy to set up once we know the sizes of the various partitions. They are indicated in (4.6.2):

$$\begin{array}{l}\\ \text{number of retained branch voltages} \\ \text{number of constitutive equations} \\ \text{number of nodes on } I\text{-graph}\end{array}\begin{array}{c}\begin{array}{ccc}\text{number of retained branch voltages} & \text{number of retained branch currents} & \text{number of nodes on } V\text{-graph}\end{array}\\ \left[\begin{array}{ccc}\mathbf{1} & \mathbf{0} & -\mathbf{A}_v^t \\ \mathbf{Y}_b & \mathbf{Z}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_i & \mathbf{0}\end{array}\right]\end{array}\begin{bmatrix}\mathbf{V}_b \\ \mathbf{I}_b \\ \mathbf{V}_n\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{W}_b \\ \mathbf{0}\end{bmatrix}. \qquad (4.6.2)$$

EXAMPLE 4.6.1. Draw the I- and V-graphs for the network of Fig. 4.1.4. Write the $\mathbf{A}_i$, $\mathbf{A}_v$ matrices, determine the sizes of the submatrices in (4.6.2), and write the two-graph tableau equation.

The graphs are in Fig. 4.6.2. Collapsing of the nodes followed the instructions given in Fig. 4.6.1. The matrices are

$$\mathbf{A}_i = \begin{array}{c|cccc} & 2 & 3 & 4 & 5 \\ \hline \boxed{1} & -1 & 1 & 1 & 0 \\ \boxed{2} & 0 & -1 & 0 & 1 \end{array}$$

$$\mathbf{A}_v = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \triangle 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \triangle 2 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ \triangle 3 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ \triangle 4 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}$$

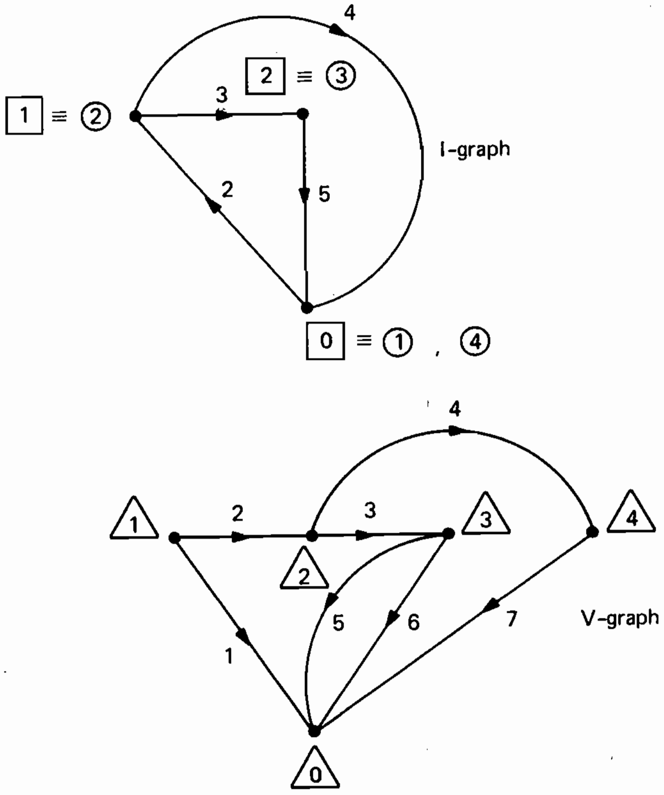


Fig. 4.6.2. Current and voltage graphs for the network in Fig. 4.1.4.

The constitutive equations are

$$V_1 = E_1$$

$$G_2V_2 - I_2 = 0$$

$$G_3V_3 - I_3 = 0$$

$$sC_4V_4 - I_4 = 0$$

$$sC_5V_5 - I_5 = 0$$

$$\mu V_6 - V_7 = 0.$$

The sizes of the submatrices can now be determined. Horizontally they are 7 + 4 + 4, vertically 7 + 6 + 2. Once the sizes are known, it is easier to fill the constitutive equations directly, one by one, without preparing the matrices. The result is shown in Fig. 4.6.3.

$$\left[\begin{array}{ccccccc|cccc|cccc}
1 & & & & & & & & & & & -1 & & & \\
 & 1 & & & & & & & & & & -1 & 1 & & \\
 & & 1 & & & & & & & & & & -1 & 1 & \\
 & & & 1 & & & & & & & & & -1 & & 1 \\
 & & & & 1 & & & & & & & & & -1 & \\
 & & & & & 1 & & & & & & & & -1 & \\
 & & & & & & 1 & & & & & & & & -1 \\
\hline
1 & & & & & & & & & & & & & & \\
 & G_2 & & & & & & -1 & & & & & & & \\
 & & G_3 & & & & & & -1 & & & & & & \\
 & & & sC_4 & & & & & & -1 & & & & & \\
 & & & & sC_5 & & & & & & -1 & & & & \\
 & & & & & \mu & -1 & & & & & & & & \\
\hline
 & & & & & & & -1 & 1 & 1 & & & & & \\
 & & & & & & & & -1 & & 1 & & & &
\end{array}\right]
\left[\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ \hline I_2 \\ I_3 \\ I_4 \\ I_5 \\ \hline V_{\triangle 1} \\ V_{\triangle 2} \\ V_{\triangle 3} \\ V_{\triangle 4}
\end{array}\right]
=
\left[\begin{array}{c}
\\ \\ \\ \\ \\ \\ \\ E_1 \\ \\ \\ \\ \\ \\ \\ \\
\end{array}\right]$$

Fig. 4.6.3. Two-graph tableau for the network in Fig. 4.1.4.

EXAMPLE 4.6.2. Prepare the two-graph tableau formulation of the generalized impedance converter of Fig. 4.1.5.

The network is redrawn, and its graphs are shown in Fig. 4.6.4. The matrices

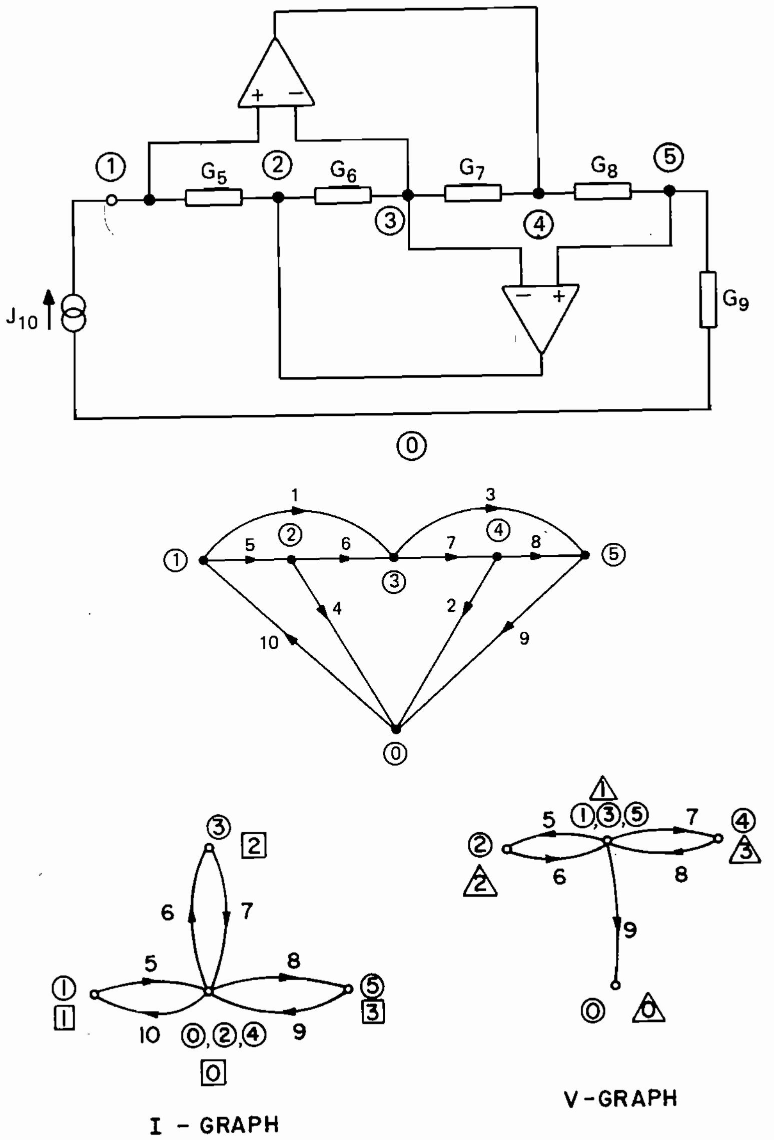


Fig. 4.6.4. Current and voltage graphs for the network of Example 4.6.2.

are

$$
\mathbf{A}_i = \begin{array}{c} \\ \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array}
\begin{array}{c}
\begin{array}{cccccc} 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

$$
\mathbf{A}_v = \begin{array}{c} \\ \triangle_1 \\ \triangle_2 \\ \triangle_3 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}
\end{array}.
$$

In the two-graph method, the OPAMPs do not have any constitutive equations. The remaining ones are

$$G_i V_i - I_i = 0, \qquad \text{for } i = 5, 6, 7, 8, 9$$

$$I_{10} = J_{10}$$

The sizes of the submatrices will be horizontally 5 + 6 + 3, vertically 5 + 6 + 3. The tableau formulation is shown in Fig. 4.6.5.

4.7. REPRESENTATION OF THE GRAPHS ON THE COMPUTER

This section describes one way of representing the graphs on the computer. The topology of the network must be given to the computer in the form of a table. Such a table will contain information on the type of the element and the nodes from which and to which the element goes. The values are not needed for this explanation.

Consider the network in Fig. 4.1.4. A table of the following form will contain the necessary information:

						VVT	
Element:	E_1	G_2	G_3	C_4	C_5	Input	Output
From node:	1	1	2	2	3	3	4
To node:	0	2	3	4	0	0	0

This representation contains the required information for the one-graph method.

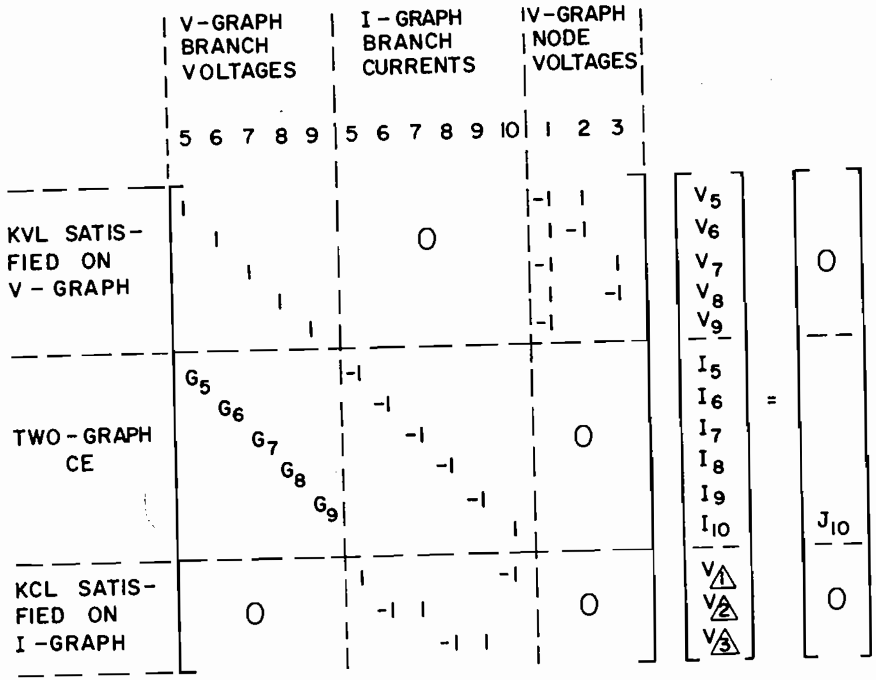


Fig. 4.6.5. Two-graph tableau for the network in Fig. 4.1.5.

In the two-graph method, two new tables must be prepared, each taking into account the collapsing of the nodes due to some elements or the absence of the edges for others.

Consider first the *I*-graph for the preceding table. The voltage source collapses nodes 0 and 1. Thus 1 changes *everywhere* in the table into 0. Moreover, the VVT collapses the output node to ground. This means that 4 will change everywhere into 0. The table for the *I*-graph thus far would be

Edge:	1	2	3	4	5	6	7
From node:	0	0	2	2	3	3	0
To node:	0	2	3	0	0	0	0

As there is no node 1, all the numbers are decreased by one. This amounts to the renumbering of the nodes on the *I*-graph:

Edge:	1	2	3	4	5	6	7
From node:	0	0	1	1	2	2	0
To node:	0	1	2	0	0	0	0

In the final step, note that there is no longer any current associated with the voltage source or with the output of the VVT and that the input current of the VVT is zero. Scan the data again and implicitly delete these edges:

Edge:	~~1~~	2	3	4	5	~~6~~	~~7~~
From node:	0	0	1	1	2	0	0
To node:	0	1	2	0	0	0	0

I-graph (4.7.1)

The *I*-graph will have two nodes and edges 2, 3, 4, 5. Edges 1, 6, 7 will be absent (they form self-loops from 0 to 0 node). The table is in agreement with the *I*-graph in Fig. 4.6.2.

For the voltage graph, one proceeds with the same basic information. Checking Fig. 4.6.1, we conclude that no collapsing of nodes will take place and all edges will be present; the graph is thus represented by the information in the original table, and we have

Edge:	1	2	3	4	5	6	7
From node:	1	1	2	2	3	3	4
To node:	0	2	3	4	0	0	0

V-graph (4.7.2)

4.8. MODIFIED NODAL FORMULATION USING *I*- AND *V*-GRAPHS

Elimination of all branch voltages and of some branch currents was considered in previous sections. The same basic approach can be applied when using the two-graph modified nodal method.

The elimination of unwanted variables is done systematically by:

1. Replacing all branch currents of elements which have admittance description by their constitutive equations. This introduces branch voltages for these elements into the KCL equations.
2. Replacing all branch voltages by the node voltages of the *V*-graph.
3. Collecting these and the remaining equations into a matrix.

The above steps are best seen on a simple example in which we eliminate the variables by using the above steps. Consider the network in Fig. 4.8.1 and its *I*- and *V*-graphs. The node numbering on the *I*-graph has been changed.

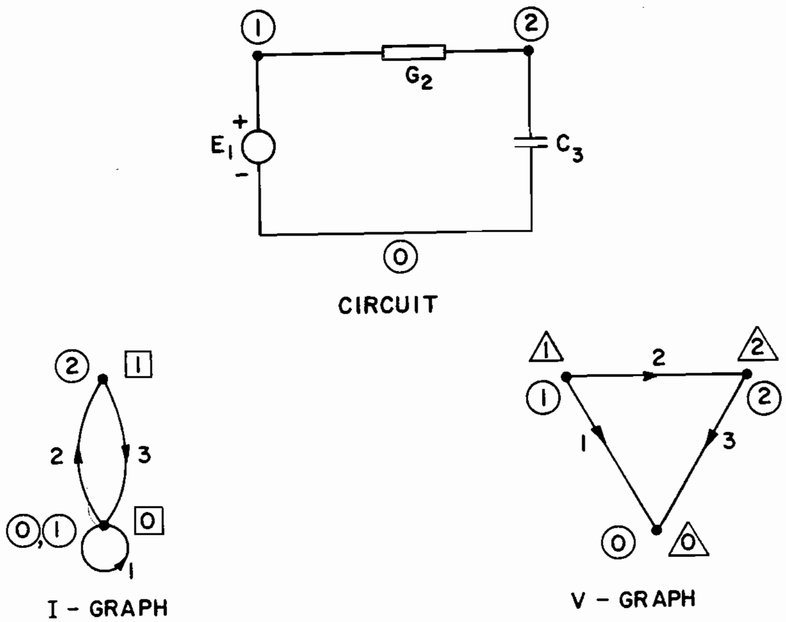


Fig. 4.8.1. Circuit demonstrating the two-graph modified nodal formulation.

There remains only one node with a nonzero number. The KCL equation is:

KCL: $$-I_2 + I_3 = 0. \tag{4.8.1}$$

From the *V*-graph we have the following branch-node voltage relations:

KVL: $$\begin{aligned} V_1 &= V_{\triangle 1} \\ V_2 &= V_{\triangle 1} - V_{\triangle 2} \\ V_3 &= V_{\triangle 2}. \end{aligned} \tag{4.8.2}$$

and the constitutive equations of the elements are

$$\begin{aligned} V_1 &= E_1 \\ I_2 &= G_2 V_2 \\ I_3 &= sC_3 V_3. \end{aligned} \tag{4.8.3}$$

Note that the current of the voltage source has been eliminated directly by the topological properties of the I-graph.

Substitution of the constitutive equations for the conductor and capacitor into the KCL leads to

$$-G_2V_2 + sC_3V_3 = 0. \tag{4.8.4}$$

Next, replace all branch voltages in (4.8.4) and in $V_1 = E_1$ by the V-graph node voltages (4.8.2). The equations, written in matrix form, become

$$\text{KCL}\begin{bmatrix} -G_2 & G_2 + sC_3 \\ \hdashline 1 & 0 \end{bmatrix}\begin{bmatrix} V_{\triangle_1} \\ V_{\triangle_2} \end{bmatrix} = \begin{bmatrix} 0 \\ E_1 \end{bmatrix} \tag{4.8.5}$$

The admittance portion of (4.8.5) has one row and two columns. The first row expresses the KCL relationship. The second row takes care of the constitutive equation of the voltage source whose current has been eliminated topologically.

Formally, the two-graph modified nodal formulation can be derived by partitioning appropriately the constitutive equations. Partitioning of the KCL and KVL equations must follow the same sequence. Subscripts on voltages refer to the V-graph, subscripts on currents to the I-graph, and the branch numbering on the two graphs is done independently. We have

$$\begin{aligned}
\mathbf{I}_1 &= \mathbf{Y}_1\mathbf{V}_1 && \text{(admittances)}\\
\mathbf{V}_2 &= \mathbf{Z}_2\mathbf{I}_2 && \text{(impedances)}\\
\mathbf{I}_3 &= \mathbf{J}_s && \text{(current sources)}\\
\mathbf{V}_3 &= \mathbf{E}_s && \text{(voltage sources)}\\
\mathbf{I}_4 &= \boldsymbol{\alpha}\mathbf{I}_5 \quad \text{(CCTs)}, && \mathbf{V}_4 = \boldsymbol{\mu}\mathbf{V}_5 \quad \text{(VVTs)}\\
\mathbf{Y}_6\mathbf{V}_6 + \mathbf{Z}_6\mathbf{I}_6 &= \mathbf{W} && \text{(general multiterminal networks).}
\end{aligned} \tag{4.8.6}$$

Since there are six types of constitutive equations, $\mathbf{A}_i$, $\mathbf{A}_v$ must be partitioned into six submatrices. The KCL equation $\mathbf{A}_i\mathbf{I}_b = \mathbf{0}$ becomes

$$\mathbf{A}_{i_1}\mathbf{I}_1 + \mathbf{A}_{i_2}\mathbf{I}_2 + \mathbf{A}_{i_3}\mathbf{I}_3 + \mathbf{A}_{i_4}\mathbf{I}_4 + \mathbf{A}_{i_5}\mathbf{I}_5 + \mathbf{A}_{i_6}\mathbf{I}_6 = \mathbf{0}. \tag{4.8.7}$$

The KVL equation $\mathbf{V}_b = \mathbf{A}^t_v\mathbf{V}_{\triangle_n}$ is rewritten as six equations:

$$\mathbf{V}_k = \mathbf{A}^t_{vk}\mathbf{V}_{\triangle_n}, \qquad k = 1, 2, 3, 4, 5, 6. \tag{4.8.8}$$

Substitute for $\mathbf{I}_1$, $\mathbf{I}_3$, and $\mathbf{I}_4$ into (4.8.7), using the constitutive equations. This results in

$$\mathbf{A}_{i_1}\mathbf{Y}_1\mathbf{V}_1 + \mathbf{A}_{i_2}\mathbf{I}_2 + (\mathbf{A}_{i_4}\boldsymbol{\alpha} + \mathbf{A}_{i_5})\mathbf{I}_5 + \mathbf{A}_{i_6}\mathbf{I}_6 = -\mathbf{A}_{i_3}\mathbf{J}_s. \tag{4.8.9}$$

In the second step, replace the branch voltage $\mathbf{V}_1$ by the first equation in (4.8.8). This results in

$$\mathbf{A}_{i_1}\mathbf{Y}_1\mathbf{A}^t_{v1}\mathbf{V}_{\triangle_n} + \mathbf{A}_{i_2}\mathbf{I}_2 + (\mathbf{A}_{i_4}\boldsymbol{\alpha} + \mathbf{A}_{i_5})\mathbf{I}_5 + \mathbf{A}_{i_6}\mathbf{I}_6 = -\mathbf{A}_{i_3}\mathbf{J}_s. \tag{4.8.10}$$

Use the remaining equations (4.8.8) and substitute them into the remaining constitutive equations (4.8.6). The result is

$$\begin{aligned}
\mathbf{A}^t_{v2}\mathbf{V}_{\triangle_n} &= \mathbf{Z}_2\mathbf{I}_2\\
\mathbf{A}^t_{v3}\mathbf{V}_{\triangle_n} &= \mathbf{E}_s\\
\mathbf{A}^t_{v4}\mathbf{V}_{\triangle_n} &= \boldsymbol{\mu}\mathbf{A}^t_{v5}\mathbf{V}_{\triangle_n}\\
\mathbf{Y}_6\mathbf{A}^t_{v6}\mathbf{V}_{\triangle_n} + \mathbf{Z}_6\mathbf{I}_6 &= \mathbf{W}.
\end{aligned} \tag{4.8.11}$$

Equations (4.8.10) and (4.8.11) can be written in matrix form as follows:

$$\begin{array}{c|cccc}
 & \text{V-graph node voltages} & \multicolumn{3}{c}{\text{subset of branch currents on I-graph}} \\
\text{KCL on I-graph} & \mathbf{A}_{i_1}\mathbf{Y}_1\mathbf{A}^t_{v1} & \mathbf{A}_{i_2} & (\mathbf{A}_{i_4}\boldsymbol{\alpha} + \mathbf{A}_{i_5}) & \mathbf{A}_{i_6} \\
 & \mathbf{A}^t_{v2} & -\mathbf{Z}_2 & \mathbf{0} & \mathbf{0} \\
\text{CE of various elements} & \mathbf{A}^t_{v3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & (\mathbf{A}^t_{v4} - \boldsymbol{\mu}\mathbf{A}^t_{v5}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & \mathbf{Y}_6\mathbf{A}^t_{v6} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_6
\end{array}
\begin{bmatrix} \mathbf{V}_{\triangle_n} \\ \mathbf{I}_2 \\ \mathbf{I}_5 \\ \mathbf{I}_6 \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{i_3}\mathbf{J}_s \\ \mathbf{0} \\ \mathbf{E}_s \\ \mathbf{0} \\ \mathbf{W} \end{bmatrix}. \tag{4.8.12}$$

This is the formal expression of the formulation with two graphs. It is *not used* in this form for actual network formulation. Its properties are studied and the conclusions applied below.

Equation (4.8.12) presents one important result: the portion $\mathbf{A}_{i_1}\mathbf{Y}_1\mathbf{A}^t_{v1}$ has a form in which *the rows satisfy the KCL equations on the I-graph*, whereas *the* *columns are expressed by nodal voltages on the V-graph.* This is the same form as in the previous type of the modified nodal formulation, the only exception being that the numbers of the rows and columns are those of the nodes given on the two independent graphs. Let an admittance y have an edge going from j_i to j_i' on the *I*-graph. Similarly, let the edge on the *V*-graph go from j_v to j_v'. The symbolic formula for entering an admittance into the nodal portion has the following form:

$$\begin{array}{llcc} & & \multicolumn{2}{c}{V\text{-graph edge pointing}} \\ & & \text{from node} & \text{to node} \\ & & j_v & j_v' \\ I\text{-graph edge pointing} & \begin{array}{l}\text{from node } j_i \\ \text{to node } j_i'\end{array} & \multicolumn{2}{c}{\begin{bmatrix} y & -y \\ -y & y \end{bmatrix}.} \end{array} \qquad (4.8.13)$$

If j_i or j_i' is zero, the row is omitted in the matrix. Similarly, if j_v or j_v' is zero, the column is absent from the matrix. As an example, consider the *I*-graph information provided by (4.7.1) and the *V*-graph information given by (4.7.2). The third element, G_3, has its *I*-graph edge pointing from node 1 to node 2, while the *V*-graph edge points from node 2 to node 3. Using the above schematic representation, $+G_3$ will be in the positions (1, 2) and (2, 3), while $-G_3$ will be in the positions (2, 2) and (1, 3). All elements not having admittance description are entered into the remaining partitions. This can be done systematically, and Fig. 4.8.2 collects all usually encountered ideal network elements and the way they are entered into the matrix of the two-graph modified nodal matrix *without writing the matrices* $\mathbf{A}_i$, $\mathbf{A}_v$. This method cannot avoid entirely the use of graphs (or their computer equivalents), but examples given below show that writing the matrix equation is as easy as in the previous cases.

EXAMPLE 4.8.1. Write the two-graph modified nodal formulation for the network of Fig. 4.1.4 without generating the matrices $\mathbf{A}_i$, $\mathbf{A}_v$. The graphs were given in Fig. 4.6.2, and their tabular equivalents are (4.7.1) and (4.7.2).

The *I*-graph has two ungrounded nodes, while the *V*-graph has four. Thus the nodal portion of the formulation will be 2 × 4.

Element 1 does not enter the matrix since it forms a self-loop on the *I*-graph. The same will be true for the input and output of the VVT (edges 6 and 7). The remaining elements are filled by the schematic rule given above. The constitutive equation for the voltage source is $V_{\triangle 1} = E_1$, and for the VVT, $V_{\triangle 4} -$

$\mu V_{\triangle 3} = 0$. They are appended below the nodal portion. The result is

$$\begin{array}{cc} & \begin{array}{cccc} \triangle 1 & \triangle 2 & \triangle 3 & \triangle 4 \end{array} \\ \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \\ \\ \end{array} & \left[\begin{array}{cccc} -G_2 & G_2 + G_3 + sC_4 & -G_3 & -sC_4 \\ 0 & -G_3 & G_3 + sC_5 & 0 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 1 \end{array}\right] \begin{bmatrix} V_{\triangle 1} \\ V_{\triangle 2} \\ V_{\triangle 3} \\ V_{\triangle 4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_1 \\ 0 \end{bmatrix} \end{array}$$

EXAMPLE 4.8.2. Consider the generalized impedance converter of Fig. 4.1.5 and its *I*- and *V*-graphs (Fig. 4.6.4). The information can be transferred into tables as follows:

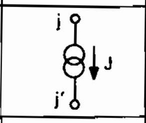
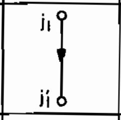
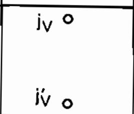
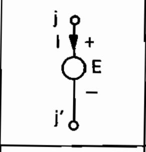
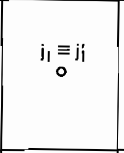
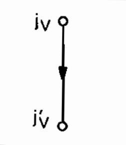
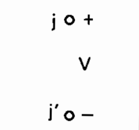
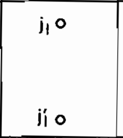
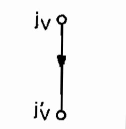
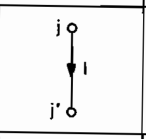
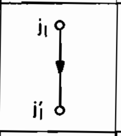
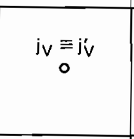
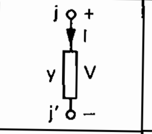
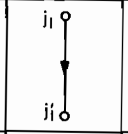
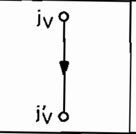
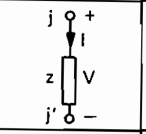
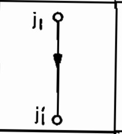
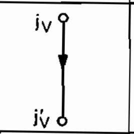
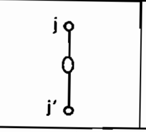
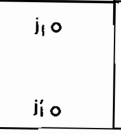
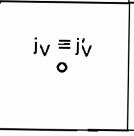
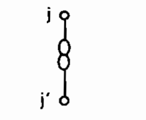
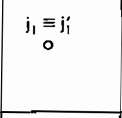
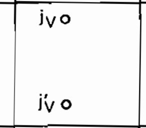
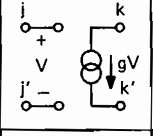
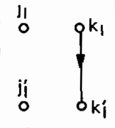
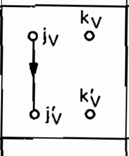
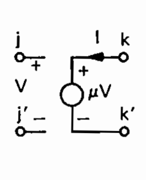
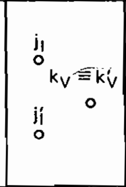
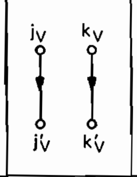
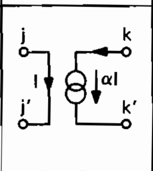
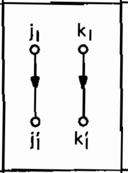
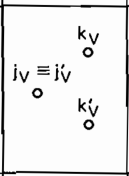
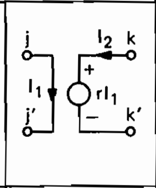
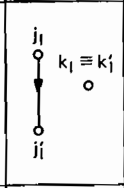
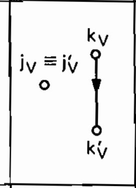
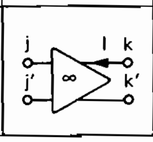
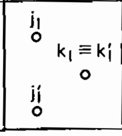
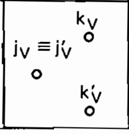
	Edge:	1	2	3	4	5	6	7	8	9	10
I-graph:	From j_i:	0	0	0	0	1	0	2	0	3	0
	To j_i':	0	0	0	0	0	2	0	3	0	1

	Edge:	1	2	3	4	5	6	7	8	9	10
V-graph:	From j_v:	0	0	0	0	1	2	1	3	1	0
	To j_v':	0	0	0	0	2	1	3	1	0	0

Both *I*- and *V*-graphs have three nodes, the matrix will be 3 × 3, and no additional constitutive equations will be appended. The result, filled by the scheme given above, is

$$\begin{array}{cc} & V\text{-graph nodes} \\ & \begin{array}{ccc} \triangle 1 & \triangle 2 & \triangle 3 \end{array} \\ I\text{-graph nodes} \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{3} \end{array} & \begin{bmatrix} G_5 & -G_5 & 0 \\ G_6 + G_7 & -G_6 & -G_7 \\ G_8 + G_9 & 0 & -G_8 \end{bmatrix} \begin{bmatrix} V_{\triangle 1} \\ V_{\triangle 2} \\ V_{\triangle 3} \end{bmatrix} = \begin{bmatrix} J_{10} \\ 0 \\ 0 \end{bmatrix} \end{array}$$

which is the same result obtained in Example 4.5.5.

ELEMENT	SYMBOL	I-GRAPH	V-GRAPH	MATRIX	EQUATIONS
CURRENT SOURCE				$\begin{array}{c} j_I \\ j'_I \end{array}\begin{bmatrix} -J \\ +J \end{bmatrix}$ SOURCE VECTOR	$I_{j_I} = J$ $I_{j'_I} = -J$
VOLTAGE SOURCE				$\begin{array}{ccc} & j_V & j'_V \\ m+1 & 1 & -1 \end{array}$ SOURCE VECTOR $\begin{bmatrix} E \end{bmatrix}$	$V_{j_V} - V_{j'_V} = E$
OPEN CIRCUIT				—	$V = V_{j_V} - V_{j'_V}$
SHORT CIRCUIT				$\begin{array}{cc} & I \\ j_I & 1 \\ j'_I & -1 \end{array}$	I IS ARBITRARY
ADMITTANCE				$\begin{array}{ccc} & j_V & j'_V \\ j_I & y & -y \\ j'_I & -y & y \end{array}$	$I_{j_I} = y(V_{j_V} - V_{j'_V})$ $I_{j'_I} = -y(V_{j_V} - V_{j'_V})$
IMPEDANCE				$\begin{array}{cccc} & j_V & j'_V & I \\ j_I & & & 1 \\ j'_I & & & -1 \\ m+1 & 1 & -1 & -z \end{array}$	$V_{j_V} - V_{j'_V} - zI = 0$ $I_{j_I} = -I_{j'_I} = I$
NULLATOR				—	—
NORATOR				—	—
VCT				$\begin{array}{ccc} & j_V & j'_V \\ k_I & g & -g \\ k'_I & -g & g \end{array}$	$I_{k_I} = g(V_{j_V} - V_{j'_V})$ $I_{k'_I} = -g(V_{j_V} - V_{j'_V})$
VVT				$\begin{array}{ccccc} & j_V & j'_V & k_V & k'_V \\ m+1 & -\mu & \mu & 1 & -1 \end{array}$	$V_{k_V} - V_{k'_V} -$ $\mu(V_{j_V} - V_{j'_V}) = 0$
CCT				$\begin{array}{cc} & I \\ j_I & 1 \\ j'_I & -1 \\ k_I & \alpha \\ k'_I & -\alpha \end{array}$	$I_{j_I} = I$ $I_{j'_I} = -I$ $I_{k_I} = \alpha I$ $I_{k'_I} = -\alpha I$
CVT				$\begin{array}{cccc} & k_V & k'_V & I_1 \\ j_I & & & 1 \\ j'_I & & & -1 \\ m+1 & 1 & -1 & -r \end{array}$	$V_{k_V} - V_{k'_V} - rI_1 = 0$
OPERATIONAL AMPLIFIER				—	—

EXAMPLE 4.8.3. We wish to know what the idealized properties of the network in Fig. 4.8.3 are. To do so, replace the transistors by their idealized representation by means of nullators and norators. This is also done in Fig. 4.8.3, along with the I- and V-graphs. The information in the graphs can be transferred into the following tables:

	Element:	C_a	G_b	C_c
I-graph:	From j_i:	1	2	2
	To j_i':	0	0	0

	Element:	C_a	G_b	C_c
V-graph:	From j_v:	0	2	1
	To j_v':	2	0	0

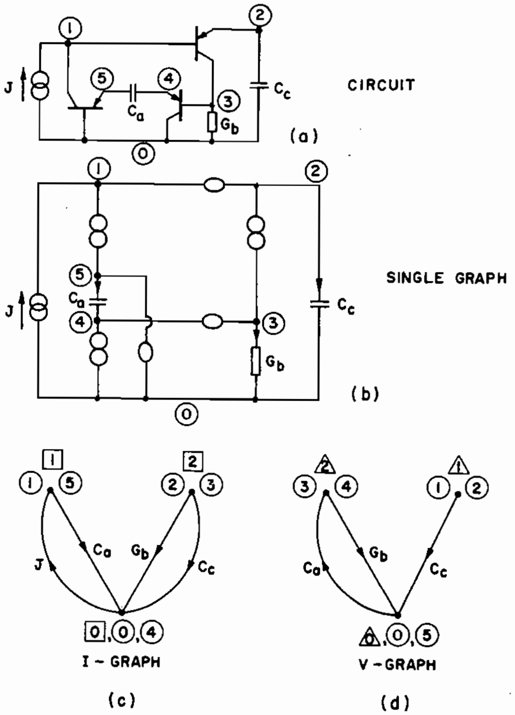


Fig. 4.8.3. (a) Transistor realization of an impedance converter; (b) idealized simulation using nullators and norators (see Fig. 1.6.6); (c, d) current and voltage graphs.

The 2×2 matrix equation is filled by inspection:

$$\begin{bmatrix} 0 & -sC_a \\ sC_c & G_b \end{bmatrix} \begin{bmatrix} V_{\triangle 1} \\ V_{\triangle 2} \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix}.$$

Solution indicates that $Z_{\text{in}} = V_1/J = G_b/s^2C_aC_c$.

4.9. SUMMARY OF THE FORMULATION METHODS

Five methods of formulating network equations were presented in this chapter. Four of them are intended for computer use, one for hand calculation.

The methods were demonstrated repeatedly in two examples: a second-order active network (Fig. 4.1.4) and a generalized impedance converter (Fig. 4.1.5). The matrix sizes and density of the matrices are compared in Tables 4.9.1 and 4.9.2.

The tableau matrices are large even for very small problems. They are always very sparse, and sparse solvers are a necessity. Unfortunately, since the matrices do not have regular structures, the renumbering and preprocessing are compli-

TABLE 4.9.1. Second-Order Active Network of Fig. 4.1.4.

	Matrix Size	Nonzero Entries	Density
One-graph tableau	18 × 18	39	12.04%
Two-graph tableau	15 × 15	33	14.67%
Modified nodal	6 × 6	15	41.67%
Two-graph modified nodal	4 × 4	9	56.25%
By hand	2 × 2	4	100%

TABLE 4.9.2. Generalized Impedance Converter of Fig. 4.1.5.

	Matrix Size	Nonzero Entries	Density
One-graph tableau	25 × 25	57	9.12%
Two-graph tableau	14 × 14	31	15.81%
Modified nodal	7 × 7	19	38.78%
Two-graph modified nodal	3 × 3	7	77.78%
By hand	3 × 3	7	77.78%

cated. Modified nodal formulations are much more compact and can be solved without sparse matrix solvers even in case of moderate-sized networks.

The two nodal formulations are recommended for problems connected with programming. The tableau equations are mainly of theoretical importance. The two-graph nodal formulation is especially advantageous for the analysis of switched networks, as demonstrated in Chapters 15 and 16.

4.10. EXAMPLE

This section will demonstrate the design of an active ninth-order Cauer-parameter low-pass filter with the following specifications: pass-band from 0 to 3470 Hz, 0.03 dB ripple, and stop-band starting at 3800 Hz with minimum attenuation of 50 dB.

The design method will be briefly explained. First, an *LC* equivalent filter is designed. The specifications are tightened to 0.02 dB in the pass-band to provide a safety margin. The filter is shown in Fig. 4.10.1, scaled to $R = 1$ and the pass-band scaled to $\omega = 1$ rad/sec. Active realization is based on impedance transformations which convert each inductor into a resistor, each resistor into a capacitor, and each capacitor into an active element called a *frequency-dependent negative resistance* (*FDNR*) (see Fig. 4.10.2). The element is realized by means of the circuit shown in Fig. 4.1.5 by replacing G_5 and G_9 by capacitances. Additional information on FDNRs and filter design using these elements can be found, for instance, in [3, 4].

The formulation method of Section 4.4 is used. Each ideal OPAMP is incorporated into the system matrix, as given by Fig. 4.4.1. The pass-band response is shown in Fig. 4.10.3 and the stop-band response in Fig. 4.10.4. A slight departure from equiripple behavior is due to the rounding of element values. This filter will be optimized in Chapter 19 with linear models of nonideal OPAMPs, and the final element values will be changed.

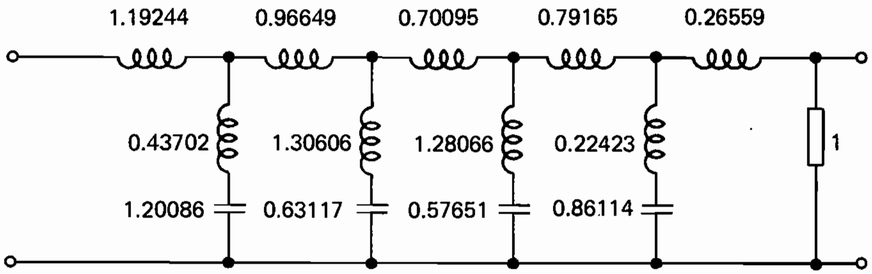


Fig. 4.10.1. Initial ninth-order Cauer-parameter low-pass filter with approximately 0.02 dB in the pass-band and minimum 50 dB attenuation in the stop band. The values are in F (farads), H (henrys), and Ω and the pass-band is scaled to $\omega = 1$ rad/sec.

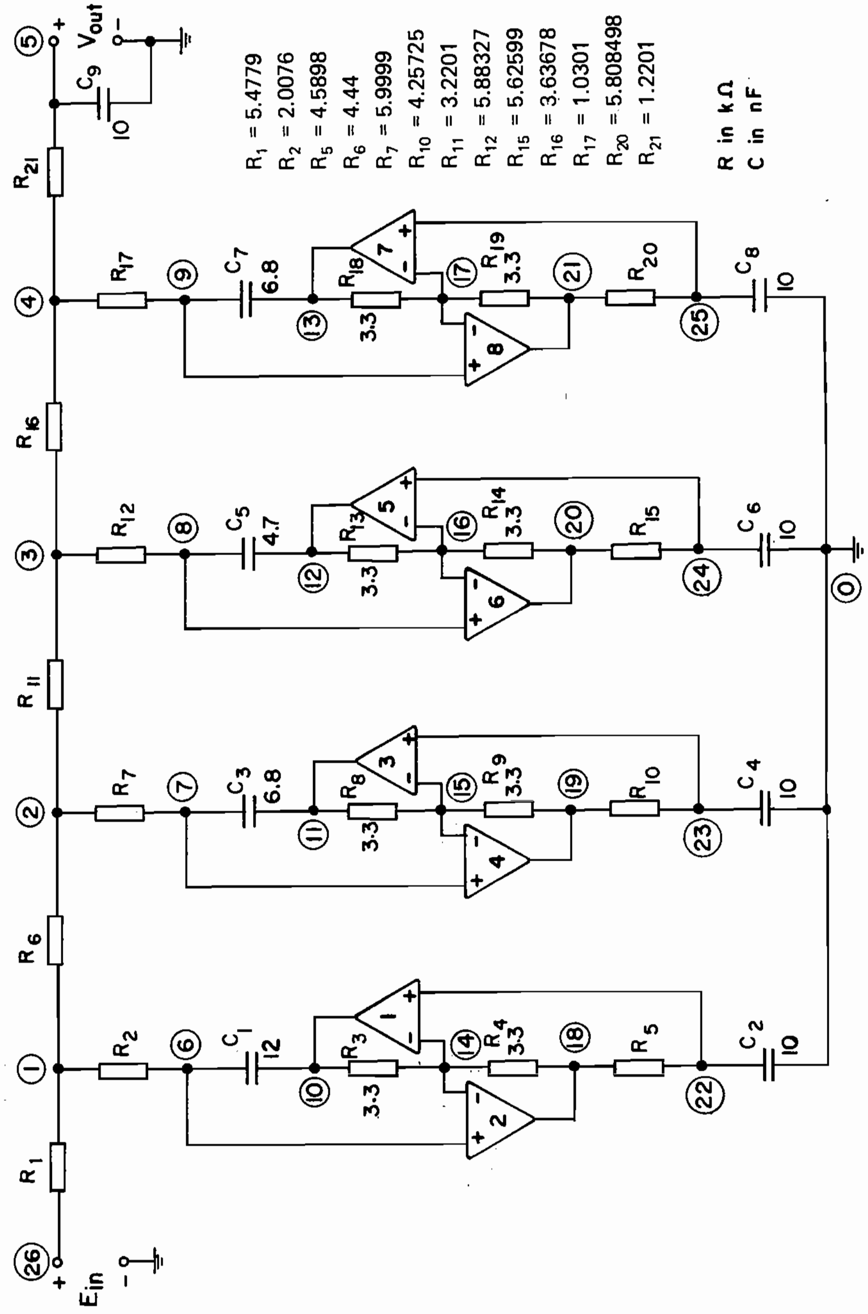


Fig. 4.10.2. Active realization of the filter shown in Fig. 4.10.1.

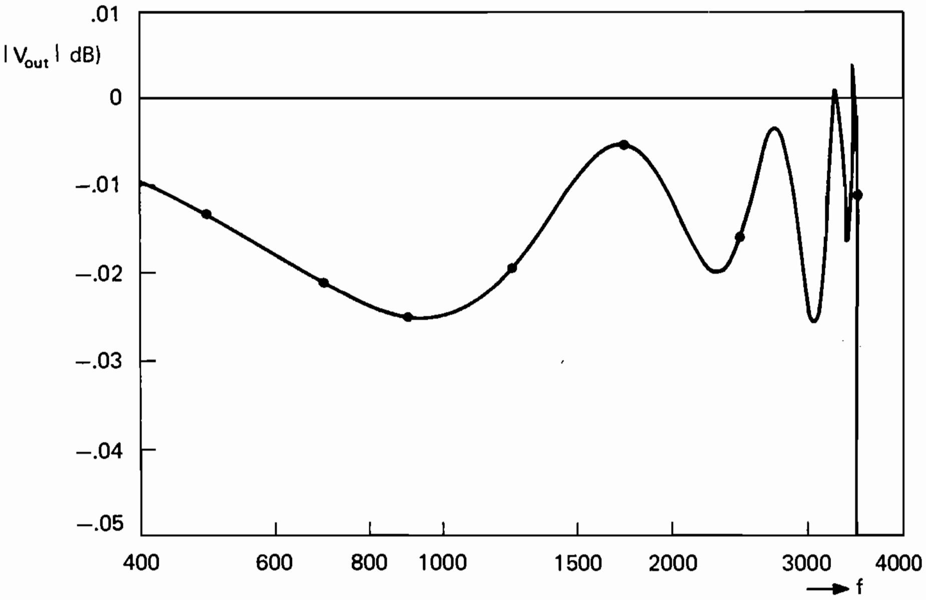


Fig. 4.10.3. Pass-band response of the filter shown in Fig. 4.10.2.

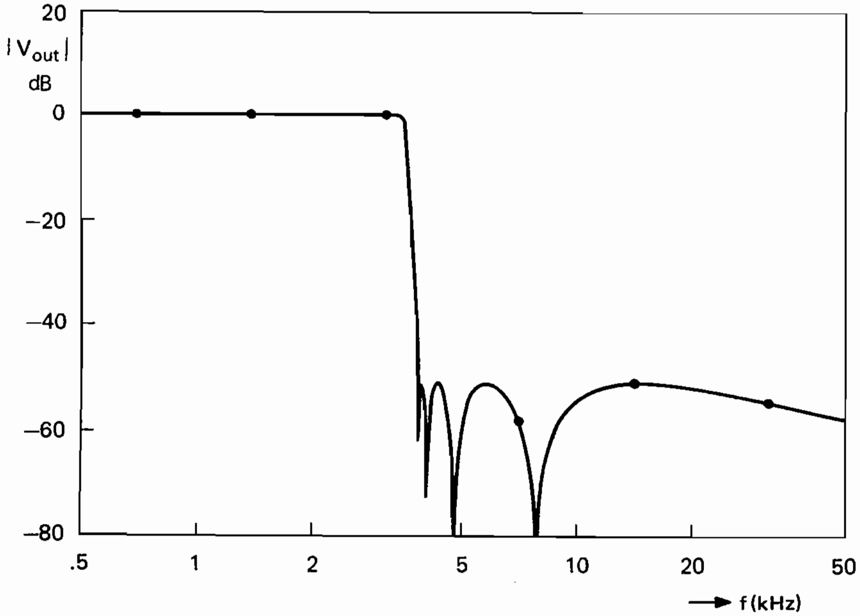


Fig. 4.10.4. Overall response of the filter in Fig. 4.10.2.

PROBLEMS

P.4.1. Write the incidence matrices for the networks shown in Fig. P.4.1 and use them in setting up the tableau matrix equations.

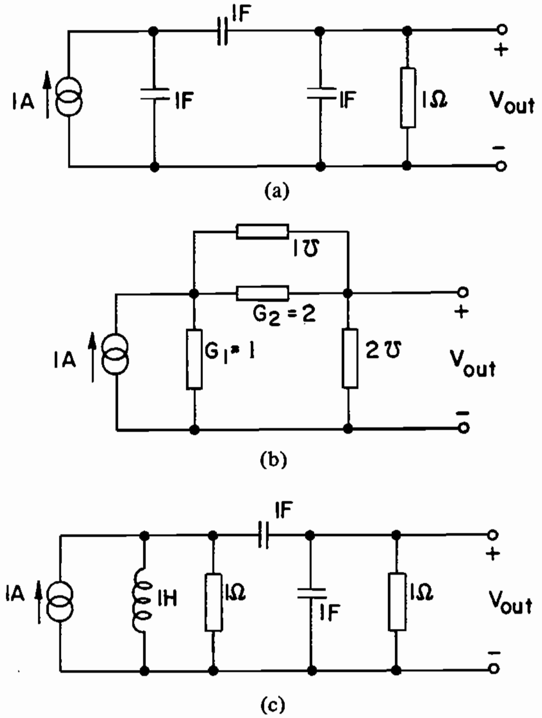


Fig. P.4.1.

P.4.2. Write the one-graph incidence matrices for the networks in Fig. P.4.2 and write their tableau matrix equations.

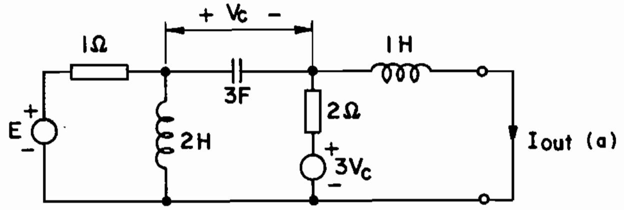
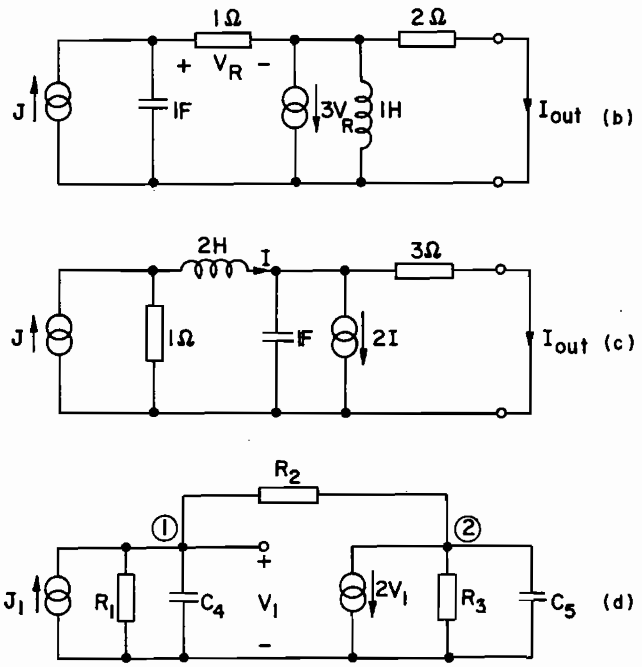


Fig. P.4.2.



(b)

(c)

(d)

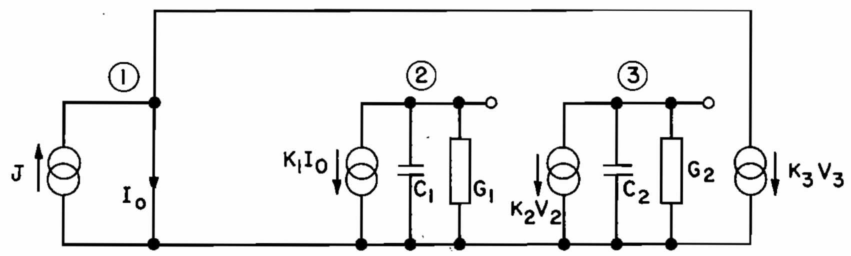
Fig. P.4.2. (*Continued*)

P.4.3. Apply the formulation of Section 4.3 to the networks shown in Fig. P.4.1.

P.4.4. Apply the modified nodal formulation without graphs to the networks shown in Fig. P.4.1.

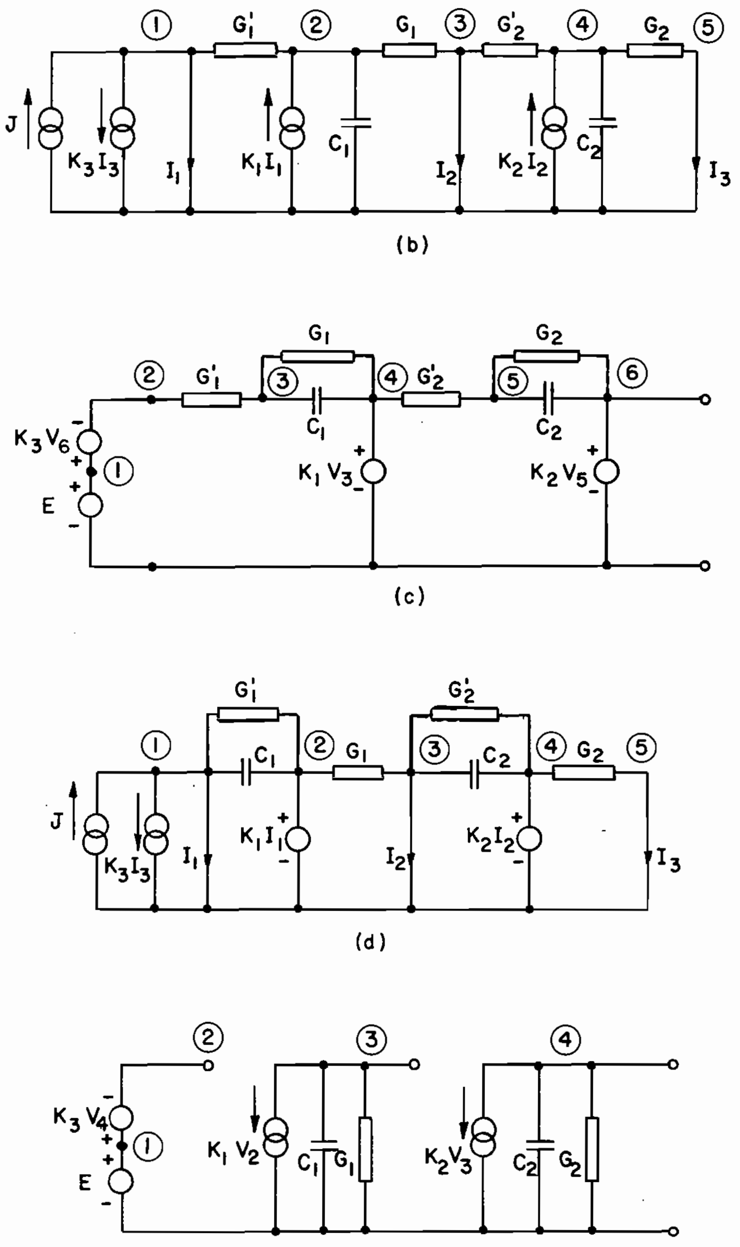
P.4.5. Apply the modified nodal formulation without graphs to the networks shown in Fig. P.4.2.

P.4.6. Practice the modified nodal formulation without graphs on the networks shown in Fig. P.4.6.



(a)

Fig. P.4.6.



(b)

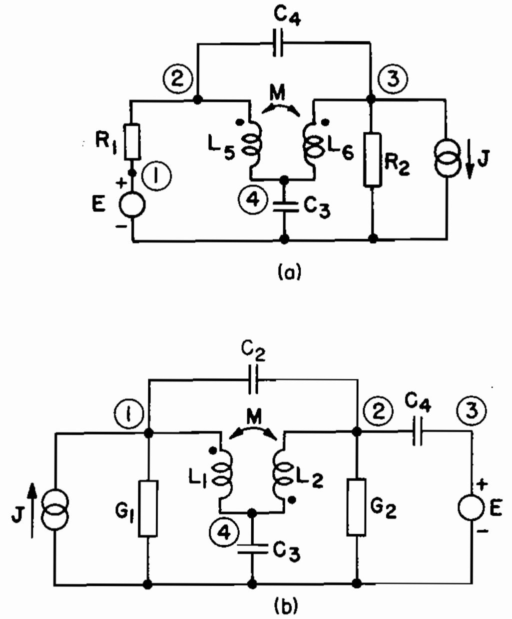
(c)

(d)

(e)

Fig. P.4.6. (*Continued*)

P.4.7. Apply the modified nodal formulation without graphs to the transformer networks shown in Fig. P.4.7.

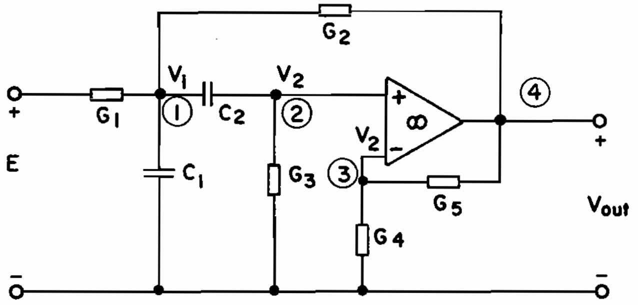


(a)

(b)

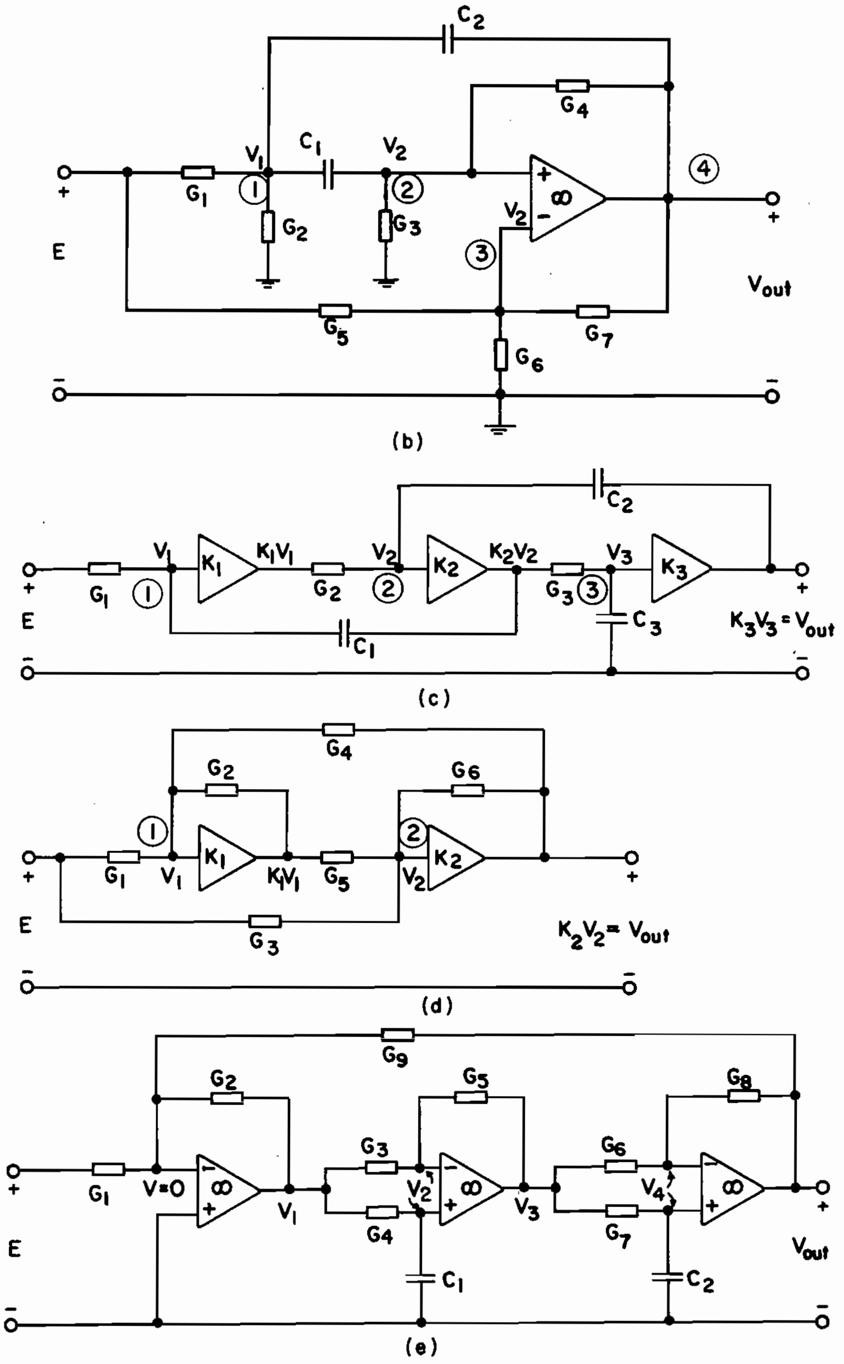
Fig. P.4.7.

P.4.8. Apply the nodal formulation of Section 4.5 to the active networks shown in Fig. P.4.8.



(a)

Fig. P.4.8.



(b)

(c)

(d)

(e)

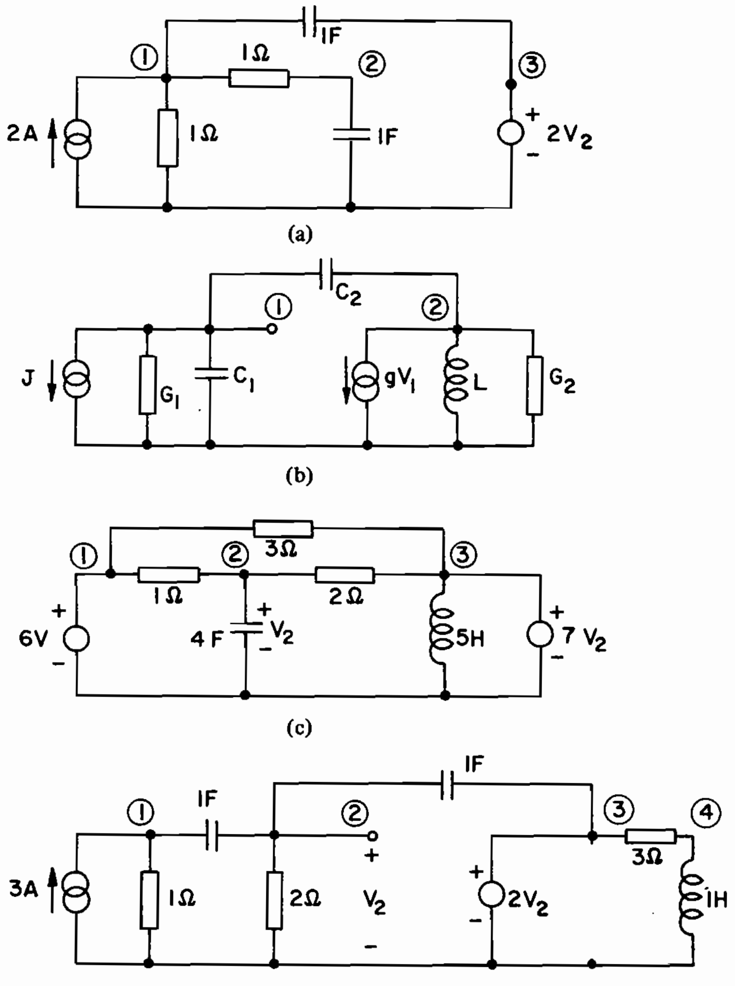
Fig. P.4.8. *(Continued)*

P.4.9. Draw separate voltage and current graphs for the networks shown in Fig. P.4.2.

P.4.10. Write the two-graph tableau formulation for the networks shown in Fig. P.4.2, using the graphs of Problem P.4.9.

P.4.11. Write the two-graph modified nodal formulation for the networks shown in Fig. P.4.2.

P.4.12. Apply the two-graph modified nodal formulation to the networks shown in Fig. P.4.12.



(a)

(b)

(c)

(d)

Fig. P.4.12.

P.4.13. The network shown in Fig. P.4.13 is a phase-shift oscillator, provided that the gain of the inverting amplifier is adjusted to $A = 29$. In such cases, the oscillation frequency is

$$\omega = \frac{G}{C}\frac{1}{\sqrt{6}} \text{ rad/sec.}$$

(a) Set up the system equations for this network using the formulation method of Section 4.5. Note that the right-hand-side vector will be a zero vector.

(b) Find the determinant of the system matrix, set it equal to zero, and substitute $s = j\omega$. Separating real and imaginary parts, obtain the condition on the gain and the oscillating frequency (given above).

(c) Find the poles of the determinant by retaining the variable s.

(d) Calculate the pole positions for $27 < A < 31$ and $G/C = 1$.

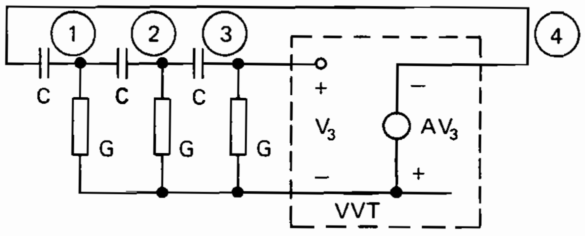


Fig. P.4.13.

P.4.14. Write the modified nodal formulation for the network in Fig. P.4.14. Enter the inductor in impedance form and use Table 4.1.1 to handle the initial conditions.

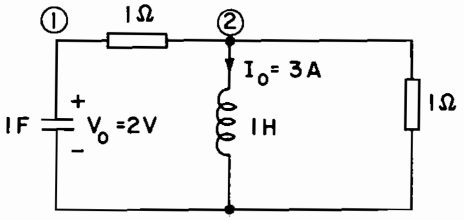


Fig. P.4.14.

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1. G. D. Hachtel, R. K. Brayton, and F. G. Gustavson: The sparse tableau approach to network analysis and design. *IEEE Transactions on Circuit Theory*, Vol. CT-18, pp. 101–113, January 1971.
2. C. W. Ho, A. E. Ruehli, and P. A. Brennan: The modified nodal approach to network analysis. *IEEE Transactions on Circuits and Systems*, Vol. CAS-22, pp. 504–509, June 1975.
3. L. T. Bruton: *RC-Active Circuits: Theory and Design*. Prentice-Hall, Englewood Cliffs, NJ, 1980.
4. L. P. Huelsman and P. E. Allen: *Introduction to the Theory and Design of Active Filters*. McGraw-Hill, New York, 1980.

5

Sensitivities

This chapter introduces the concept of sensitivity and provides basic information by considering network functions in explicit form. Computational aspects of sensitivity evaluation will be covered in Chapter 6.

Sensitivities are mathematical measures that provide additional insight into the behavior of a physical system. There are three main reasons for their study:

1. Sensitivities help in the understanding of how variations of parameters, such as those of element values, influence the response.
2. They help in comparing the quality of various networks having the same nominal response.
3. They provide response gradients in optimization applications.

Various sensitivity definitions are introduced in Section 5.1 and applied to the most common response variables: network functions, their poles and zeros, and the Q's and ω's of the poles and zeros. The formulae derived are valid for networks of arbitrary complexity.

As a network response is usually influenced by simultaneous variations in several parameters, multiparameter sensitivity is discussed and its use demonstrated in Section 5.2.

The designer is often interested in the behavior of his network in the presence of parasitic elements. Under ideal conditions, these elements have zero nominal values. Sensitivities are defined with respect to these elements and can be used to predict response variations when parasitics take on small values. Section 5.3