# Introduction to Options Pricing Theory

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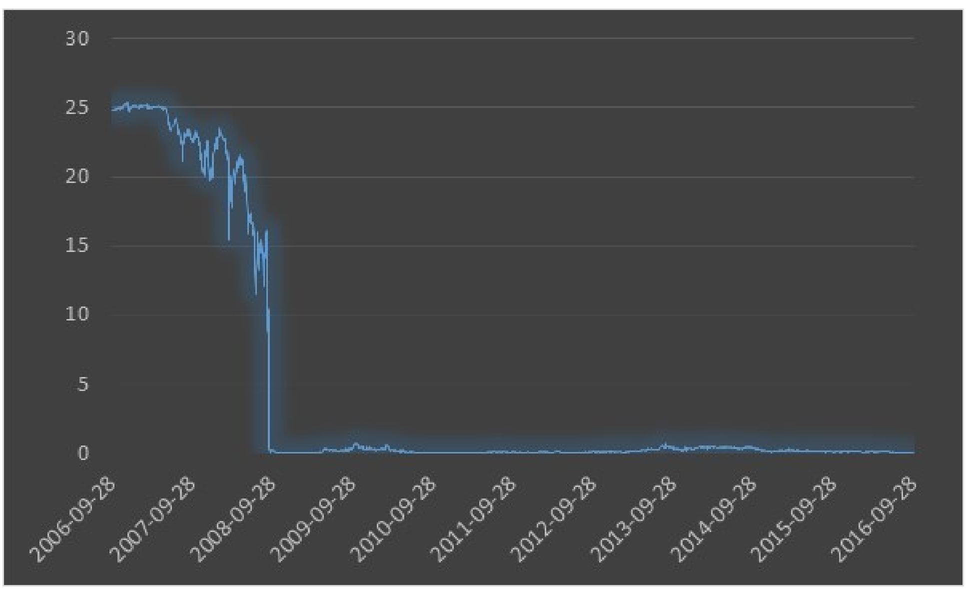


# Preface

This text presents a self-contained introduction to the binomial model and the Black-Scholes model in options pricing theory. It is the main literature for the course "Options and Mathematics" at Chalmers, which provides the students with a first rudimentary knowledge in mathematical finance (in particular, without using stochastic calculus). The pre-requisites to follow this text are the standard basic courses in mathematics, such as calculus and linear algebra. No previous knowledge on probability theory and finance are required. Each chapter is complemented with a number of exercises and Matlab codes. The exercises marked with the symbol (?) aim to critical thinking and do not necessarily have a well-defined unique solution. The solution of the exercises marked with the symbol (•) and the answer to those marked with the symbol (⋆) can be found in appendixes B and C at the end of the text. Further exercises are found in Appendix D. Finally the proof of the theorems marked with the symbol (∗) can be skipped on first reading.

**Remark:** The Matlab codes presented in this text are not optimized. Moreover the powerful vectorization tools of Matlab are not employed, in order to make the codes easily adaptable to other computer softwares and languages. The task to improve the codes presented in this text is left to the interested reader.

**Front cover picture:** 10 years historical price of the Lehman Brothers stock

# Contents

# Chapter 1

# Warm-up

The purpose of this chapter is threefold: (1) introduce a few basic financial concepts, (2) formulate and discuss the main assumptions behind the standard theory of options pricing, (3) derive some fundamental qualitative properties of option prices.

## 1.1 Basic financial concepts

For a more detailed discussion on the concepts introduced in this section, see [2].

### Financial assets

The term **asset** may be used to identify any economic resource capable of producing value and which, under specific legal terms, can be bought and sold (i.e., converted into cash). Assets may be tangible (e.g., lands, buildings, commodities, etc.) or intangible (e.g., patents, copyrights, stocks, etc.). Assets are also divided into **real assets**, i.e, assets whose value is derived by an intrinsic property (e.g., tangible assets), and **financial assets**, such as stocks, options, bonds, etc., whose value is instead derived from a contractual claim on the income generated by another (possibly real) asset. For example, upon holding shares of the Volvo stock (a financial asset), we can make a profit from the production and sale of cars even if we do not own an auto plant (a real asset). As we consider only financial assets in these notes, the terms “asset” and “financial asset” will be henceforth used interchangeably.

### Price

The **price** of a financial asset is the value, measured in some units of currency (e.g. dollars), at which the **buyer** and the **seller** agree to exchange ownership of the asset. The price is chosen by the two parties as a result of some kind of “negotiation”. More precisely, the **ask**

**price** is the minimum price at which the seller is willing to sell the asset, while the **bid price** is the maximum price that the buyer is willing to pay for the asset. When the difference between these two values, called **bid-ask spread**, becomes zero, the exchange of the asset takes place at the corresponding price.

A generic financial asset will be denoted by $\mathcal{U}$ and its price at time $t$ by $\Pi^{\mathcal{U}}(t)$. Prices are generally positive, although some financial assets (e.g., forward contracts) have zero price.

The asset price refers to the price of a **share** of the asset, where "share" stands for the minimum amount of an asset which can be traded. In these notes all prices are given in a fixed currency, which is however left unspecified.

## Markets

Financial assets can be traded in **official** markets or in **over the counter (OTC)** markets. In the former case all trades are subject to a common legislation, while in the latter the exchange conditions are agreed upon by the individual traders. Example of OTC markets are the currency markets (Forex) and the bond markets, while stock markets, option markets and futures markets are all examples of official markets. Buyers and sellers of assets in a market will be called **investors** or **agents**.

## Long and short position

Besides the usual operations of "buying" and "selling" the asset, we need to consider an additional common type of transaction, which is called **short-selling**. Short-selling an asset (typically a stock) is the practice of selling the asset without actually owning it. Concretely, an investor is short-selling $N$ shares of an asset if the investor borrows the shares from a third party and then sell them immediately on the market. The reason for short-selling an asset is the expectation that the price of the asset will decrease in the future. More precisely, assume that $N$ shares of an asset $\mathcal{U}$ are short-sold at time $t = 0$ for the price $\Pi^{\mathcal{U}}(0)$ and let $T > 0$ be the time at which the shares must be returned to their original owner. If $\Pi^{\mathcal{U}}(T) < \Pi^{\mathcal{U}}(0)$, then upon re-purchasing the $N$ shares at time $T$, and returning them to the lender, the short-seller will make the profit $N(\Pi^{\mathcal{U}}(0) - \Pi^{\mathcal{U}}(T))$.

An investor is said to have a **long position** on an asset if the investor owns the asset and will therefore profit from an increase of the price of the asset. Conversely, the investor is said to have a **short position** on the asset if the investor will profit from a decrease of its value, as it happens for instance when the investor is short-selling the asset.

Finally we remark that any transaction in the market is subject to **transaction costs** (e.g., broker's commissions and lending fees for short-selling) and **transaction delays** (trading in real markets is not instantaneous).

## Stocks and dividends

The **capital stock** of a company is the part of the company equity capital that is made publicly available for trading. Stocks are most commonly traded in official markets (**stock markets**). For instance, over 300 company stocks are traded in the Stockholm exchange market. The price per share at time $t > 0$ of a generic stock will be denoted by $S(t)$.

A stock may occasionally pay a **dividend** to its shareholders. This means that a fraction of the stock price is deposited to the bank account of the shareholders. The day at which the dividend is paid, as well as its amount in percentage of the opening stock price at this day, are known in advance. After the dividend has been paid, the price of the stock usually drops of the amount paid by the dividend.

## Market index and ETFs

A market **index** is a weighted average of the value of a collection of assets traded in one or more official markets. For example, S&P500 (Standard and Poor 500) measures the average value of 500 stocks traded at the New York stock exchange (NYSE) and NASDAQ markets. Market indexes can be regarded themselves as assets. More precisely an **ETF** (Exchange Traded Fund) on a market index is a financial asset whose value tracks exactly the value of a market index (or a given fraction thereof). Hence one share of an ETF on S&P500 will increase its value of 1% in one day if during that day S&P 500 has gained 1%. An **inverse ETF** however will in the same example decrease its value of 1%. Thus ETFs give investors the possibility to speculate whether the market will gain or loose value in the future.

## Portfolio position and portfolio process

Consider an agent that invests on $N$ assets $\mathcal{U}_1, \dots, \mathcal{U}_N$ during the time interval $[0, T]$. Assume that the agent trades on $a_1$ shares of the asset $\mathcal{U}_1$, $a_2$ shares of the asset $\mathcal{U}_2$,…, $a_N$ shares of the asset $\mathcal{U}_N$. Here $a_i \in \mathbb{Z}$, where $a_i < 0$ means that the investor has a short position in the asset $\mathcal{U}_i$, while $a_i > 0$ means that the investor has a long position in the asset $\mathcal{U}_i$ (the reason for this interpretation will become soon clear). The vector $\mathcal{A} = (a_1, a_2, \dots, a_N) \in \mathbb{Z}^N$ is called a **portfolio position**, or simply a portfolio. The **value** of the portfolio at time $t$ is given by

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i \Pi^{\mathcal{U}_i}(t), \quad t \in [0, T], \tag{1.1}$$

where $\Pi^{\mathcal{U}_i}(t)$ denotes the price of the asset $\mathcal{U}_i$ at time $t$. The value of the portfolio measures the wealth of the investor: the higher is $V(t)$, the "richer" is the investor at time $t$. Now we see that when the price of the asset $\mathcal{U}_i$ increases, the value of the portfolio increases if $a_i > 0$ and decreases if $a_i < 0$, which explains why $a_i > 0$ corresponds to a long position on the asset $\mathcal{U}_i$ and $a_i < 0$ to a short position. We also remark that portfolios can be added by using the linear structure on $\mathbb{Z}^N$, namely if $\mathcal{A}, \mathcal{B} \in \mathbb{Z}^N$, $\mathcal{A} = (a_1, \dots, a_N)$, $\mathcal{B} = (b_1, \dots, b_N)$ are two portfolios and $\alpha, \beta \in \mathbb{Z}$, then $\mathcal{C} = \alpha\mathcal{A} + \beta\mathcal{B}$ is the portfolio $\mathcal{C} = (\alpha a_1 + \beta b_1, \dots, \alpha a_N + \beta b_N)$.

In the definition of portfolio position and portfolio value given above, the investor keeps the same number of shares of each asset during the whole time interval $[0, T]$. Suppose now that the investor changes the position on the assets at some times

$$0 = t_0 < t_1 < t_2 < \dots < t_M = T;$$

for simplicity we assume that at each time $t_1, \dots t_M$ the change in the portfolio position occurs instantaneously. Let $\mathcal{A}_0$ denote the initial (at time $t = t_0 = 0$) portfolio position of the investor and $\mathcal{A}_j$ denote the portfolio position of the investor in the interval of time $(t_{j-1}, t_j]$, $j = 1, \dots, M$. As positions hold for one instance of time only are clearly meaningless, we may assume that $\mathcal{A}_0 = \mathcal{A}_1$, i.e., $\mathcal{A}_1$ is the portfolio position in the closed interval $[0, t_1]$. The vector $(\mathcal{A}_1, \dots, \mathcal{A}_M)$ is called a **portfolio process**. If we denote by $a_{ij}$ the number of shares of the asset $i$ in the portfolio $\mathcal{A}_j$, then we see that a portfolio process is in fact equivalent to the $N \times M$ matrix $A = (a_{ij})$, $i = 1, \dots, N$, $j = 1, \dots, M$. The value $V(t)$ of the portfolio process at time $t$ is given by the value of the corresponding portfolio position at time $t$ as defined by (1.1). Hence for $t \in (t_{j-1}, t_j]$ and $j = 1, \dots, M$ the value of the portfolio process is given by

$$V(t) = V_{\mathcal{A}_j}(t) = \sum_{i=1}^{N} a_{ij} \Pi^{\mathcal{U}_i}(t).$$

The initial value $V(0) = V_{\mathcal{A}_0} = V_{\mathcal{A}_1}(0)$ of the portfolio, when it is positive, is called the **initial wealth** of the investor.

A portfolio process is said to be **self-financing** if no cash is ever withdrawn or infused in the portfolio. Let us look at one example. Suppose that at time $t_0 = 0$ the investor is short 400 shares on the asset $\mathcal{U}_1$, long 200 shares on the asset $\mathcal{U}_2$ and long 100 shares on the asset $\mathcal{U}_3$. This corresponds to the portfolio

$$\mathcal{A}_0 = (-400, 200, 100),$$

whose value is

$$V_{\mathcal{A}_0} = -400\,\Pi^{\mathcal{U}_1}(0) + 200\,\Pi^{\mathcal{U}_2}(0) + 100\,\Pi^{\mathcal{U}_3}(0).$$

If this value is positive, the investor needs an initial wealth to set up this portfolio position: the income deriving from short selling the asset $\mathcal{U}_1$ does not suffice to open the desired long position on the other two assets.

As mentioned before, we may assume that the investor keeps the same position in the interval $(0, t_1]$, i.e., $\mathcal{A}_1 = \mathcal{A}_0$. The value of the portfolio process at time $t = t_1$ is

$$V(t_1) = V_{\mathcal{A}_1}(t_1) = -400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1).$$

Now suppose that at time $t = t_1$ the investor buys 500 shares of $\mathcal{U}_1$, sells $x$ shares of $\mathcal{U}_2$, and sells all the shares of $\mathcal{U}_3$. Then in the interval $(t_1, t_2]$ the investor has a new portfolio which is given by

$$\mathcal{A}_2 = (100, 200 - x, 0),$$

and so the value of the portfolio process for $t \in (t_1, t_2]$ is given by

$$V(t) = 100\,\Pi^{\mathcal{U}_1}(t) + (200 - x)\,\Pi^{\mathcal{U}_2}(t), \quad t \in (t_1, t_2].$$

If we now take the limit of this quantity as $t \to t_1^+$, we get the value of the portfolio "immediately after" the position has been changed at time $t_1$. Calling $V(t_1+)$ this limit we have (assuming that the prices are continuous)

$$V(t_1+) = 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1).$$

The difference between the value of the two portfolios immediately after and immediately before the transaction is then

$$\begin{aligned} V(t_1+) - V(t_1) &= 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1) \\ &\quad - (-400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1)) \\ &= 500\,\Pi^{\mathcal{U}_1}(t_1) - x\,\Pi^{\mathcal{U}_2}(t_1) - 100\,\Pi^{\mathcal{U}_3}(t_1). \end{aligned}$$

If this difference is positive, then the new portfolio cannot be created from the old one without extra cash. Conversely, if this difference is negative, then the new portfolio is less valuable than the old one, the difference being equivalent to cash withdrawn from the portfolio. Hence for self-financing portfolio processes we must have $V(t_1+) - V(t_1) = 0$ (and similarly $V(t_j+) - V(t_j) = 0$, for all $j = 1, \dots M - 1$). This implies in particular that the number $x$ of shares of the asset $\mathcal{U}_2$ to be sold at time $t_1$ in a self-financing portfolio must be

$$x = \frac{500\Pi^{\mathcal{U}_1}(t_1) - 100\Pi^{\mathcal{U}_3}(t_1)}{\Pi^{\mathcal{U}_2}(t_1)}.$$

Of course, $x$ will be an integer only in exceptional cases, which means that perfect self-financing strategies in real markets are almost impossible.

The **return** of a self-financing portfolio process in the interval $[0, T]$ is given by

$$R(T) = V(T) - V(0), \tag{1.2}$$

where $V(t)$ denotes the value of the portfolio at time $t$. If the return is positive, the investor makes a **profit** in the interval $[0, T]$, if it is negative the investor incurs in a **loss**. When $V(0) > 0$ we may also compute the **relative return** of the portfolio, which is given by

$$R_*(T) = \frac{V(T) - V(0)}{V(0)}. \tag{1.3}$$

A portfolio process is non-self-financing if some cash is withdrawn or added to the portfolio at some time $t \in (0, T)$, as it happens for instance when one or more assets in the portfolio pays a dividend in the period $(0, T)$. This cash flow must be included in the computation of the return of the portfolio. Assume for instance that the investor adds the cash $C_1$ into the portfolio at time $t_1 \in (0, T)$ and withdraws the cash $C_2$ at time $t_2 \in (0, T)$. Then the return of the portfolio in the interval $[0, T]$ is $V(T) - V(0) + C_2 - C_1$.

Finally we remark that investment returns are commonly "annualized" by dividing the return $R(T)$ by the time $T$ expressed in fraction of years (e.g., $T = 1$ week $= 1/52$ years).

## Historical volatility

The historical volatility of an asset measures the amplitude of the time fluctuations of the asset price, thereby giving information on its level of uncertainty. It is computed as the standard deviation of the log-returns of the asset based on historical data. More precisely, let $[t_0, t]$ be some interval of time in the past, with $t$ denoting possibly the present time, and let $T = t - t_0 > 0$ be the length of this interval. Let us divide $[t_0, t]$ into $n$ equally long periods, say

$$t_0 < t_1 < t_2 < \dots t_n = t, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \dots n.$$

The set of points $\{t_0, t_1, \dots t_n\}$ is called a **partition** of the interval $[t_0, t]$. Assume for instance that the asset is a stock. The **log-return** of the stock price in the interval $[t_{i-1}, t_i]$ is given by[1]

$$R_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \dots n. \tag{1.4}$$

The (corrected) sample variance of the log-returns is then

$$\Delta(t) = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2,$$

where

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right) \tag{1.5}$$

is the sample mean of log-returns. To obtain the **T-historical variance** of the asset we divide $\Delta(t)$ by $h$ measured in fraction of years, that is

$$\widehat{\sigma}_T^2(t) = \frac{1}{h}\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2 \quad (T\text{-historical variance}). \tag{1.6}$$

The square root of the $T-$historical variance is the **T-historical volatility**:

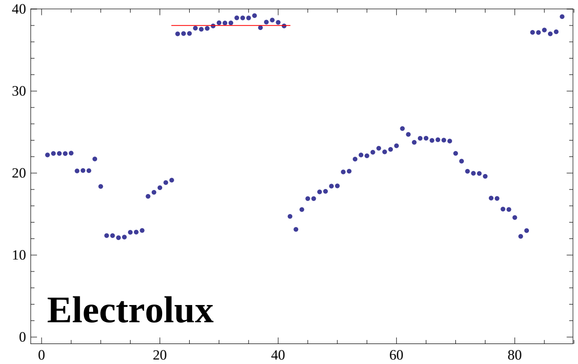
$$\widehat{\sigma}_T(t) = \frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2} \quad (T\text{-historical volatility}). \tag{1.7}$$

Note carefully that the historical volatility depends on the partition being used to compute it.

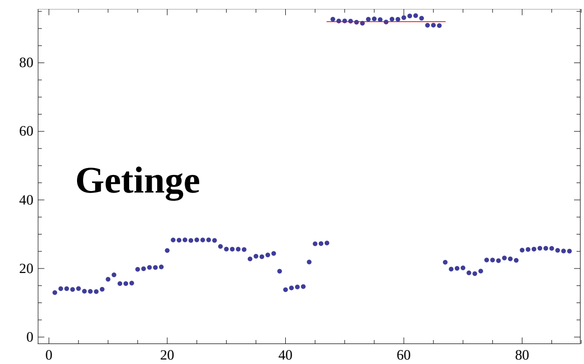
Suppose for example that $t - t_0 = T = 20$ days, which is quite common in the applications, and let $t_1, \dots t_{20}$ be the market closing times at these days. Let $h = 1$ day $= 1/365$ years. Then

$$\widehat{\sigma}_{20}(t) = \sqrt{365}\sqrt{\frac{1}{19}\sum_{i=1}^{n}(R_i - \bar{R})^2}$$

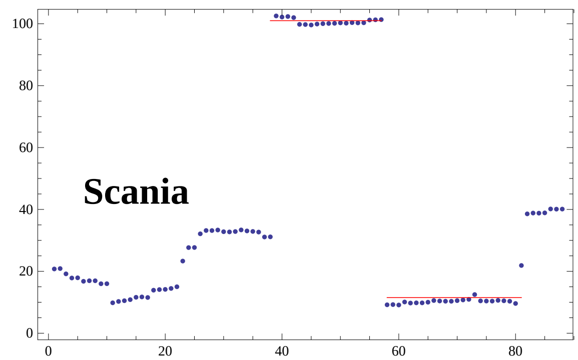
[1] Throughout these notes, $\log x$ stands for the natural logarithm of $x > 0$ (which is also frequently denoted by $\ln x$ in the literature).



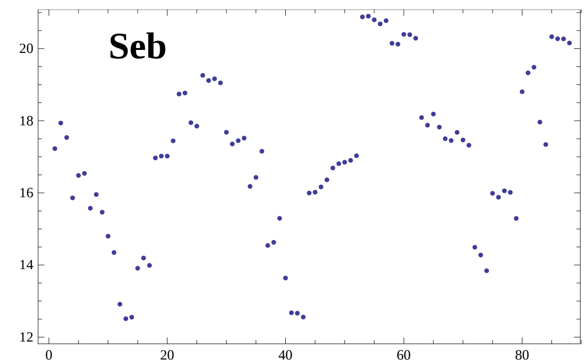

(a) ELUXY-B.ST




(b) GETIB.ST




(c) SCV-B.ST




(d) SEB-C.ST

Figure 1.1: 20-days volatility of 4 stocks in the Stockholm exchange market on May 2nd, 2014. The caption in each graph shows the ticker of the stock.

is called the 20-days historical volatility. We remark that $h = 1/252$ is also commonly used as normalization factor, since there are 252 trading days in one year.

As a way of example, Figure 1.1 shows the 20-days volatility of four stocks in the Stockholm exchange market from January 1st, 2014 until May 2nd, 2014 (88 trading days). These data have been obtained with MATHEMATICA by running the following command on May 3rd, 2014:

```
FinancialData["ticker", "Volatility20Day", {2014, 1, 1}]
```

Upon running this command, the software connects to *Yahoo Finance* and collects the 20-days volatility data for the stock identified by the ticker symbol *"ticker"*, starting from the date $\{2014, 1, 1\}$ (year, month, day) until the present day. Note that in a few cases the historical volatility remains approximately constant within periods of about 20 days.

## Financial derivatives. Options

A **financial derivative** (or derivative security) is an asset whose value depends on the performance of one (or more) other asset(s), which is called the **underlying asset**. There exist various types of financial derivatives, the most common being options, futures, forwards and swaps. In this section we discuss option derivatives on a single asset, which could be for instance a stock[2].

A **call option** is a contract between two parties: the buyer, or **owner**, of the call and the seller, or **writer**, of the call. The contract gives the owner the right, but *not* the obligation, to buy the underlying asset for a given price, which is fixed at the time when the contract is stipulated, and which is called **strike price** of the call. If the buyer can exercise this right only at some given time $T$ in the future then the call option is called **European**, while if the option can be exercised at any time earlier than or equal to $T$, then the option is called **American**. The time $T$ is called **maturity time**, or **expiration date** of the call. The writer of the call is obliged to sell the asset to the buyer if the latter decides to exercise the option. If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

In exchange for the option, the buyer must pay a **premium** to the seller (options are not free). Suppose that the option is a European option with strike price $K$ and maturity time $T$. Assume that the underlying is a stock with price $S(t)$ at time $t \leq T$ and let $\Pi_0$ be the premium paid by the buyer to the seller. In which case is it then convenient for the buyer to exercise the option at maturity? Let us define the **pay-off** of the European call as

$$Y_{\text{call}} = (S(T) - K)_+ := \max(0, S(T) - K) = \begin{cases} 0 & \text{if } S(T) \leq K \\ S(T) - K & \text{if } S(T) > K \end{cases} .$$

Similarly, we define the pay-off of the European put by

$$Y_{\text{put}} = (K - S(T))_+ = \begin{cases} 0 & \text{if } S(T) \geq K \\ K - S(T) & \text{if } S(T) < K \end{cases} .$$

Clearly, the buyer should exercise the call option at maturity if and only if $Y_{\text{call}} > 0$, as in this case it is cheaper to buy the stock at the strike price rather than at the market price. Similarly the owner of the put should exercise if and only if $Y_{\text{put}} > 0$, as in this case the income generated by selling the stock at the strike price is higher then the income generated by selling it at the market price. Hence the call or put option must be exercised at maturity if and only if the pay-off is positive, in which case the option is said to **expire in the money**. The return for the owner of the option is given by $N(Y_{\text{call}} - \Pi_0)$ in the case of the call and by $N(Y_{\text{put}} - \Pi_0)$ in the case of the put, where $N$ is the number of option contracts in the buyer

[2]Options are available on many different types of assets, including currencies, market indexes, commodities, etc.

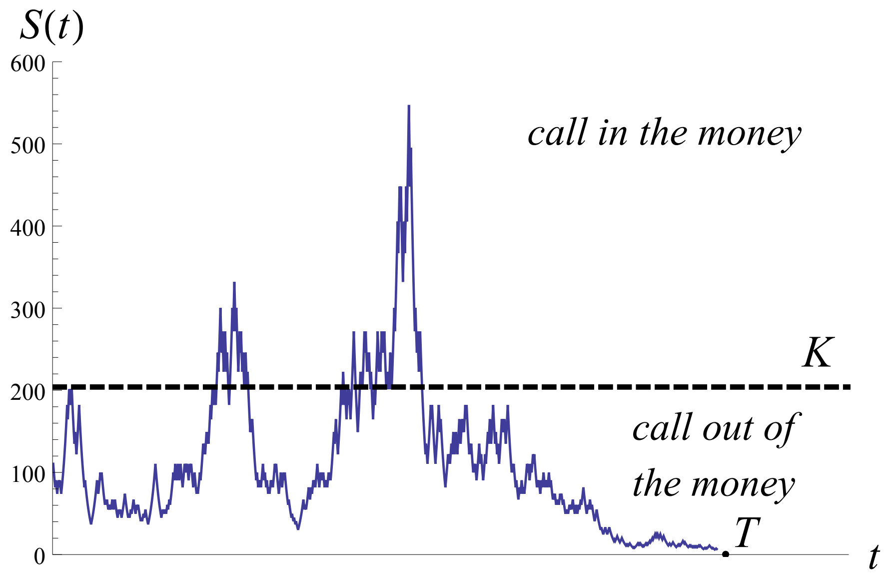



Figure 1.2: The call option with strike $K = 200$ and maturity $T$ is in the money in the upper region and out of the money in the lower region. The put option with the same parameters is in the money in the lower region and out of the money in the upper region.

portfolio[3]. Note carefully that the buyer makes a profit only if the pay-off is greater than the premium. One of the main problems in options pricing theory is to define a reasonable fair value for the price $\Pi_0$ of options (and other derivatives).

Let us introduce some further terminology. The European call (resp. put) with strike $K$ is said to be **in the money** at time $t$ if $S(t) > K$ (resp. $S(t) < K$). The call (resp. put) is said to be **out of the money** if $S(t) < K$ (resp. $S(t) > K$). If $S(t) = K$, the (call or put) option is said to be **at the money** at time $t$. The meaning of this terminology is self-explanatory, see Figure 1.2.

The pay-off of American calls exercised at time $t$ is $Y(t) = (S(t) - K)_+$, while for American puts we have $Y(t) = (K - S(t))_+$. The quantity $Y(t)$ is also called **intrinsic value** of the American option. In particular, the intrinsic value of an out-of-the-money American option is zero.

## Option markets

Option markets are relatively new compared to stock markets. The first one has been established in Chicago in 1974 (the Chicago Board Options Exchange, CBOE). In an option market anyone (after a proper authorization) can be the buyer or the seller of an option.

[3]Options are typically sold in multiples of 100 shares, hence the minimum amount of options that one can buy is 100, which cover 100 shares of the underlying asset.

Market options are available on different assets (stocks, debts, indexes, etc.) and with different strikes and maturities. Most commonly, market options are of American style.

Clearly, the deeper in the money is the option, the higher will be the price of the option in the market, while the price of an option deeply out of the money is usually quite low (but never zero!). It is also clear that the buyer of the option is the party holding the long position on the option, since the buyer owns the option and thus hopes for an increase of its value, while the writer is the holder of the short position.

One reason why investors buy call options is to protect a short position on the underlying asset. In fact, suppose that an investor is short-selling 100 shares of a stock at time $t = 0$ for the price $S(0)$ and let $t_0 > 0$ be the time at which the shares must be returned to the lender. At time $t = 0$ the investor buys 100 shares of an American call option on the stock with strike $K \approx S(0)$ and maturity later than $t_0$. If at time $t_0$ the price of the stock is no lower than $S(0)$, the investor will exercise the call and thus obtain 100 shares of the stock for the price $K \approx S(0)$. So doing the investor will be able to return the shares to the lender with minimal losses. At the same fashion, investors buy put options to protect a long position on the underlying asset[4].

**Exercise 1.1** (?). *Can you think of a reason why investors sell options?*

Of course, speculation is also an important factor in option markets. However the standard theory of options pricing is firmly based on the interpretation of options as derivative securities and does not take speculation into account.

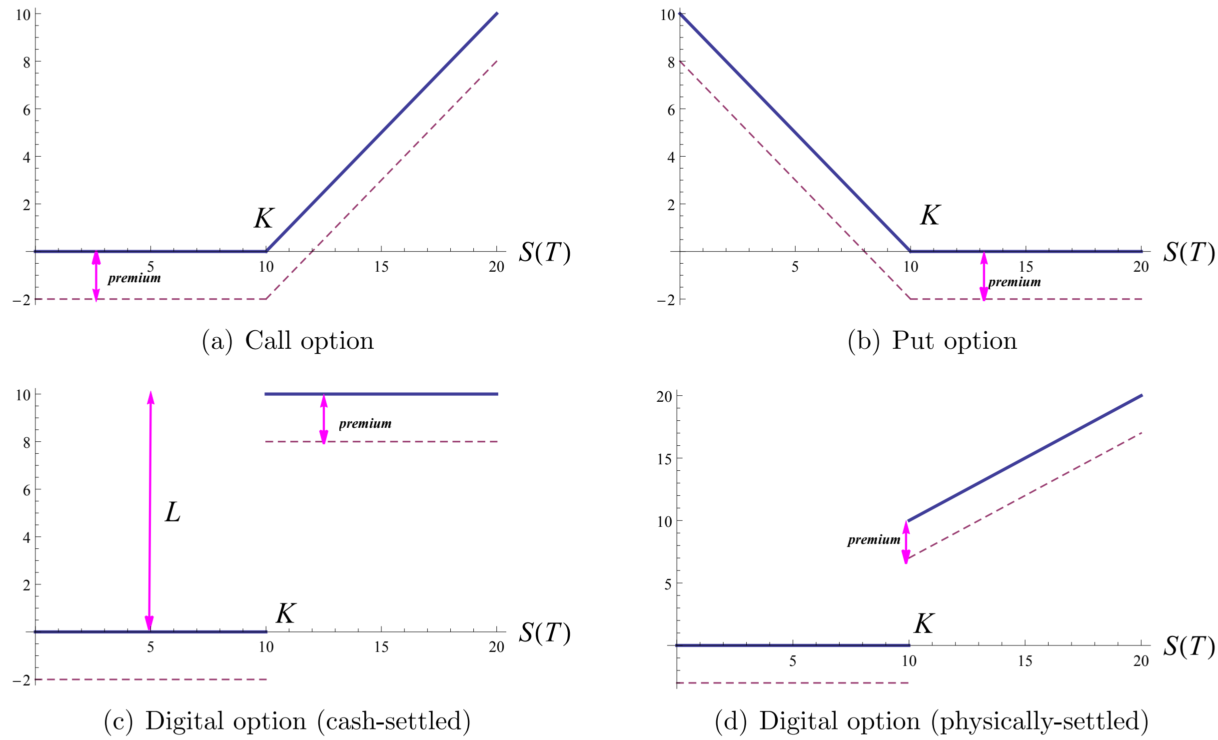
## European, American and Asian derivatives

By far the majority of financial derivatives, including options other then simple calls and puts, are traded OTC. In this section we discuss a few examples, but first it is convenient to introduce a precise mathematical definition of European and American derivatives.

Given a function $g : (0, \infty) \to \mathbb{R}$, the **standard** European derivative with pay-off $Y = g(S(T))$ and maturity time $T > 0$ is the contract that pays to its owner the amount $Y$ at time $T > 0$. Here $S(T)$ is the price of the underlying stock at time $T$, while $g$ is the **pay-off function** of the derivative (e.g., $g(x) = (x - K)_+$ for European call options, while $g(x) = (K - x)_+$ for European put options). Hence, the pay-off of standard European derivatives depends only on the price of the stock at maturity and not on the earlier history of the stock price. An example of standard European derivative which is actually traded in the market (other than call and put options) is the **digital option**. Denote by $H(x)$ the Heaviside function,

$$H(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases},$$

[4]A short-selling strategy that is not covered by a suitable security is said to be naked.




(a) Call option

(b) Put option

(c) Digital option (cash-settled)

(d) Digital option (physically-settled)

Figure 1.3: Pay-off function (continuous line) and return (dashed line) of some standard European derivatives.

and let $K, L > 0$ be constants expressed in units of some currency (e.g., dollars). The standard European derivative with pay-off function $g(x) = LH(x-K)$ is called **cash-settled digital call option**; this derivative pays the amount $L$ if $S(T) > K$, and nothing otherwise. The **physically-settled digital call option** has the pay-off function $g(x) = xH(x - K)$, which means that at maturity the buyer receives either the stock (when $S(T) > K$), or nothing. Digital options are also called **binary** options. Figure 1.3 shows the pay-off function of call, put and digital call options with strike $K = 10$. Drawing the pay-off function of a derivative helps to get a first insight onto its properties.

**Exercise 1.2.** *Given $K, \Delta K > 0$, consider the standard European derivative with maturity $T$ and pay-off function*

$$g(x) = (x - K + \Delta K)_+ - 2(x - K)_+ + (x - K - \Delta K)_+.$$

*Draw the graph of $g$ and derive the range of $S(T)$ for which the derivative expires in the money.*

If the pay-off depends on the history of the stock price during the interval $[0, T]$, and not just on $S(T)$, we shall say that the contract is a **non-standard** European derivative. An example of non-standard European derivative is the so-called **Asian call option**, the pay-off of which is given by $Y = (\frac{1}{T}\int_0^T S(t)\,dt - K)_+$.

The value at time $t$ of the European derivative with pay-off $Y$ and expiration date $T$ will be denoted by $\Pi_Y(t)$ (we do not include the expiration date in our notation).

The term "European" refers to the fact that the contract cannot be exercised before time $T$. For a **standard American derivative** the buyer can exercise the contract at any time $t \in (0, T]$ and so doing the buyer will receive the amount $Y(t) = g(S(t))$, where $g$ is the pay-off function of the American derivative. Non-standard American derivatives can be defined similarly to the European ones, but with the further option of earlier exercise.

**Exercise 1.3.** *Look for the definition of the following options: Bermuda option, Compound option, Lookback option, Barrier option, Chooser option. Classify them as American/European, standard/non-standard and write down their pay-off function.*

## Money market

A **money market** is a (OTC) market in which the objects of trading are **short term loans**, i.e., loans with maturity between one day and one year[5]. Assets in the money market bear a very low risk of default and for this reason they are also called **risk-free** assets, although this terminology is criticized by many scholars. Examples of risk-free assets in the money market are commercial papers and repurchase agreements (**repo**). In contrast to stock and option markets, money markets are typically accessible only by financial institutions and not by private investors.

The value at time $t$ of a generic risk-free asset in the money market will be denoted by $B(t)$; the difference $B(t_2) - B(t_1)$ is determined by the interest rate of the asset in the interval $[t_1, t_2]$. We say that a risk-free asset has **instantaneous interest rate** $r(t)$ in the interval $[t_1, t_2]$ if

$$B(t) = B(t_1) \exp\left(\int_{t_1}^{t} r(s)\,ds\right), \quad \text{for } t_1 \leq t \leq t_2. \tag{1.8}$$

Under normal market conditions we have $B(t_2) > B(t_1)$, i.e., the interest rate of risk-free assets is usually positive, but exceptions are possible. For instance, short term loans issued by the Swedish central bank (Riksbanken) have presently (2017) a negative interest rate.

To see how the money market works in practice, suppose that an investor buys a risk-free asset at time $t = 0$ which expires at time $T > 0$. The seller will receive the quantity $B_0 = B(0)$. As part of the agreement, the seller promises to re-purchase the risk-free asset at time $T$ for $B(T) > B_0$. Hence buying an asset in the money market is equivalent to lend money to the seller, while selling an asset in the money market is equivalent to borrow money from the buyer. The seller of the risk-free asset is the party holding the short position on the asset, while the buyer holds the long position.

[5] Loan contracts with maturity longer than one year are called **bonds** and are traded in the bond market.

We remark that there exists several ways to define the interest rate of a risk-free asset; in particular, the interest rate may be **compounded** discretely in time, rather then continuously as in (1.8). Inasmuch as in these notes we use only (1.8) to compute the value of assets in the money market, we shall refer to $r(t)$ simply as the interest rate of the risk-free asset, i.e., the word "instantaneous" will be omitted for brevity. Moreover we shall always assume that the interest rate is a constant $r$, so that (1.8) simplifies to

$$B(t) = B(t_1)\exp(r(t - t_1)), \quad \text{for } t_1 \leq t \leq t_2. \tag{1.9}$$

The interest rate is measured in yearly percentage. For example, if one share of a risk-free asset has initial value $B(0) = 10$ at time $t = 0$ and interest rate 10% per year, then after 1 month=1/12 years its value is $B(1/12) = 10\exp(0.1/12) \approx 10.0837$.

Note that so far we have introduced three strategies that investors can undertake to obtain cash: short-selling an asset, writing an option or borrowing from the money market.

## Frictionless markets

As all mathematical models, also those in options pricing theory are based on a number of assumptions. Some of these assumptions are introduced only with the purpose of simplifying the analysis of the models and often correspond to facts that do not occur in reality. Among these "simplifying" assumptions we impose that

1. There is no bid/ask spread[6]
2. There are no transaction costs and trades occur instantaneously
3. An investor can trade any fraction of shares
4. No lack of liquidity: there is no limit to the amount of cash that can be borrowed from the money market

We have seen in the previous sections that real markets do not satisfy exactly these assumptions, although in some case they do it with reasonable approximation. For instance, if the investor is an agent working for a large financial institution, then the above assumptions reflect reality quite well. However they work very badly for private investors. We summarize the validity of these assumptions by saying that the market has **no friction**. The idea is that when the above assumptions hold, trading proceeds "smoothly without resistance".

In a frictionless market we may define the portfolio process of an agent who is investing on $N$ assets during the time interval $[0, T]$ as a function

$$\mathcal{A} : [0, T] \to \mathbb{R}^N, \quad \mathcal{A}(t) = (a_1(t), \dots, a_N(t)),$$

[6] In particular, any offer to buy/sell an asset is matched by an offer to sell/buy the asset.

i.e., by assumptions 2 and 3, the number of shares $a_i(t)$ of each single asset at time $t$ is now allowed to be any real number and to change at any arbitrary time in the interval $[0,T]$. Portfolio processes can be added using the linear structure in $\mathbb{R}^N$, namely if $\mathcal{B} = (b_1(t),\dots,b_N(t))$, then $\mathcal{A}+\mathcal{B}$ is the portfolio

$$\mathcal{A}+\mathcal{B} = (a_1(t)+b_1(t),\dots,a_N(t)+b_N(t)).$$

The value at time $t$ of the portfolio process $\mathcal{A}$ is

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a(t)\Pi^{\mathcal{U}_i}(t),$$

and clearly

$$V_{\mathcal{A}}(t) + V_{\mathcal{B}}(t) = V_{\mathcal{A}+\mathcal{B}}(t).$$

Moreover it is clear that, thanks to assumption 3, perfect self-financial portfolio processes in frictionless markets always exist.

A further simplifying assumption that we make in the rest of these notes is the following:

5. All risk-free assets in the the money market have the same constant interest rate $r$

Although $r$ is usually positive in the applications, we shall sometimes allow $r=0$ or even a negative interest rate. We shall refer to $r$ as the interest rate of the money market. In the applications it is customary to choose the value of $r$ to be an **interbank offered rate**, such as LIBOR, or EURIBOR, etc., that is the average interest rate at which banks in a given geographical zone lend money to one another.

## 1.2 Qualitative properties of option prices

The purpose of this section is to derive some qualitative properties of option prices using only basic principles, without invoking any specific mathematical model for the market dynamics. The following notation will be used. $S(t)$ denotes the price at time $t>0$ of a given stock, $C(t,S(t),K,T)$ denotes the price at time $t\in[0,T]$ of the European call option on the stock with strike $K>0$ and maturity $T>0$. The price of the European put option with the same parameters will be denoted by $P(t,S(t),K,T)$; finally $\widehat{C}(t,S(t),K,T)$ and $\widehat{P}(t,S(t),K,T)$ denote the values of the corresponding American call and put option.

### No-dummy investor principle

Probably the most self-evident of all financial principles is the following, which we call the **no-dummy investor principle**[7]:

[7]More commonly (and respectfully) known as **rational investor principle**.

*Investors prefer more to less and do not undertake trading strategies which result in a sure loss.*

This principle has a number of straightforward consequences. For example, an investor will never exercise an option which is out of the money, while an option that expires in the money is always exercised[8]. Moreover the price of stocks and options (prior to expire) is always positive.

**Exercise 1.4** (?). *Use the no-dummy investor principle to justify the following properties.*

*(i) The price of a financial derivative tends to its pay-off as maturity is approached. In particular, for European call/put options,*

$$C(t, S(t), K, T) \to (S(T) - K)_+, \quad P(t, S(t), K, T) \to (K - S(T))_+,$$

*as $t \to T^-$ and similarly for American options;*

*(ii) An American derivative is at least as valuable as its European counterpart. In particular, for call/put options,*

$$\widehat{C}(t, S(t), K, T) \geq C(t, S(t), K, T), \quad \widehat{P}(t, S(t), K, T) \geq P(t, S(t), K, T)$$

*(iii) The price of an American derivative is always larger or equal to its intrinsic value. In particular, for American call/put options,*

$$\widehat{C}(t, S(t), K, T) \geq (S(t) - K)_+, \quad \widehat{P}(t, S(t), K, T) \geq (K - S(t))_+.$$

Any reasonable mathematical model for the price of options must be consistent with the properties (i)-(iii) in the previous exercise. In the rest of this section they are assumed to hold without any further comment.

## Arbitrage-free principle

An **arbitrage opportunity** is an investment strategy that requires no initial wealth and which ensures a positive profit without taking any risk. For example, suppose that at time $t = 0$ an investor sells one share the American call option with strike $K$ and maturity $T_1$ and buys one share of the American call on the same stock with the same strike but with maturity $T_2 > T_1$. Suppose that the price of the latter option is lower than the price of the former, i.e., $\widehat{C}_2 := \widehat{C}(0, S(0), K, T_2) < \widehat{C}(0, S(0), K, T_1) := \widehat{C}_1$. The investor will then have the cash $\widehat{C}_1 - \widehat{C}_2$ available to buy shares of a risk-free asset in the money market. This

[8]Provided of course the owner of the option can afford to exercise. For instance, the buyer of a call option may not have the cash required to buy the underlying when the call expires in the money.

portfolio is clearly riskless: if the buyer of the option with maturity $T_1$ decides to exercise at some time $t \leq T_1$, the investor can pay-off the buyer by exercising his/her own option. Hence this investment is an example of arbitrage opportunity: it requires no initial wealth, it entails no risk and it ensures a positive profit. However, why should the investor be able to find someone willing to pay more for an option that expires earlier? This would be of course a dummy investment for the buyer. Due to the complexity of modern markets, arbitrage opportunities do actually exist, but only for a very short time, as they are quickly exploited and "traded away" by investors.

The previous discussion leads us to assume the validity of the so-called **arbitrage-free principle:**

> *Asset prices in a market are such that no arbitrage opportunities can be found.*

Asset prices that are consistent with this principle are said to be arbitrage free or **fair**.

## Dominance principle

The arbitrage-free principle can be used to derive a number of qualitative properties of option prices, which are not as obvious as (i)-(iii) in Exercise 1.4. To this purpose we need first to express the arbitrage-free principle in a more quantitative form. There are several ways to do this, e.g. by requiring the absence of arbitrage portfolios in the market (see next chapter), or by imposing the so-called **dominance principle** [3]:

**Dominance Principle:** *Suppose that $t < T$ is the present time and consider a portfolio which does not contain dividend-paying assets or short positions on American derivatives. If the value of the portfolio is non-negative at time $T$, i.e., $V(T) \geq 0$, then $V(t) \geq 0$.*

The fact that the dominance principle must hold as a consequence of the arbitrage-free principle is clear. In fact, if $V(t) < 0$, then the investor needs no initial wealth to open the portfolio, while on the other hand the portfolio return is positive, since $V(T) - V(t) \geq -V(t) > 0$. Hence the given portfolio ensures a positive profit without taking any risk, which violates the arbitrage-free principle.

Let us comment further on the formulation of the dominance principle. First of all, the requirement that the portfolio does not contain short positions on American derivatives is necessary, otherwise there is no guarantee that the portfolio exists up to time $T$ (the buyer may exercise the derivative prior to $T$). The reason to require that the assets pay no dividend is the following. Suppose that a stock pays a dividend of 2 % at time $T$. Just before that the investor open a short position on the stock and invest 99% of the income on a risk-free asset. Hence the value of this portfolio is negative, but it becomes instantaneously positive when the dividend is paid[9].

The following simple theorem will be used for our applications of the dominance principle.

**Theorem 1.1.** *Assume that the dominance principle holds and let $\mathcal{A}$, $\mathcal{B}$ be two portfolios which do not contain dividend pay assets or American derivatives. Then, for $t < T$:*

*(a) If $V_{\mathcal{A}}(T) = 0$, then $V_{\mathcal{A}}(t) = 0$;*

*(b) If $V_{\mathcal{A}}(T) = V_{\mathcal{B}}(T)$, then $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$;*

*(c) If $V_{\mathcal{A}}(T) \geq V_{\mathcal{B}}(T)$, then $V_{\mathcal{A}}(t) \geq V_{\mathcal{B}}(t)$.*

*Proof.* (a) The dominance principle implies $V_{\mathcal{A}}(t) \geq 0$. As the portfolio $\mathcal{A}$ does not contain American derivatives, the dominance principle applies to the portfolio $-\mathcal{A}$ as well. We obtain $V_{-\mathcal{A}}(t) = -V_{\mathcal{A}}(t) \geq 0$, or $V_{\mathcal{A}}(t) \leq 0$. Hence $V_{\mathcal{A}}(t) = 0$. (b) follows by part (a) and the relation $V_{\mathcal{A}-\mathcal{B}}(t) = V_{\mathcal{A}}(t) - V_{\mathcal{B}}(t)$. The proof of (c) is similar. □

The next theorem collects a number of properties that must be satisfied by option prices as a consequence of the dominance principle. We denote these properties by (iv)-(vii) in order to continue the list (i)-(iii) given in Exercise 1.4.

**Theorem 1.2.** *Assume that the dominance principle holds and let $r$ be the interest rate of the money market. Then, for all $t < T$,*

*(iv) The* **put-call parity** *holds*

$$S(t) - C(t, S(t), K, T) = Ke^{-r(T-t)} - P(t, S(t), K, T). \tag{1.10}$$

*(v) If $r \geq 0$, then $C(t, S(t), K, T) \geq (S(t) - K)_+$; the strict inequality $C(t, S(t), K, T) > (S(t) - K)_+$ holds when $r > 0$.*

*(vi) If $r \geq 0$, the map $T \to C(t, S(t), K, T)$ is non-decreasing.*

*(vii) The maps $K \to C(t, S(t), K, T)$ and $K \to P(t, S(t), K, T)$ are convex*[10].

*Proof.* (iv) Consider a constant portfolio $\mathcal{A}$ which is long one share of the stock and one share of the put option, and is short one share of the call and $K/B(T)$ shares of the risk-free asset. The value of this portfolio at maturity is

$$V_{\mathcal{A}}(T) = S(T) + (K - S(T))_+ - (S(T) - K)_+ - \frac{K}{B(T)}B(T) = 0.$$

[9] In practice this is not a feasible strategy as the profit is extremely small and highly surpassed by transaction costs. Moreover certain markets require to own the stock for a sufficiently long period of time in order to be entitled to the next dividend.

[10] Recall that a real-valued function $f$ on an interval $I$ is convex if $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$, for all $x, y \in I$ and $\theta \in (0, 1)$.

Hence, by (a) of Theorem 1.1, $V_{\mathcal{A}}(t) = 0$, for $t < T$, that is

$$S(t) + P(t, S(t), K, T) - C(t, S(t), K, T) - Ke^{-r(T-t)} = 0,$$

which is the claim.

(v) We can assume $S(t) \geq K$, otherwise the claim is obvious (the price of a call cannot be negative). By the put-call parity, using that $P(t, S(t), K, T) \geq 0$,

$$C(t, S(t), K, T) = S(t) - Ke^{-r(T-t)} + P(t, S(t), K, T) \geq S(t) - Ke^{-r(T-t)};$$

the right hand side equals $S(t) - K$ for $r = 0$ and is strictly greater than this quantity for $r > 0$. As $S(t) - K = (S(t) - K)_+$ for $S(t) \geq K$, the claim follows.

(vi) Consider a portfolio $\mathcal{A}$ which is long one call with maturity $T_2$ and strike $K$, and short one call with maturity $T_1$ and strike $K$, where $T_2 > T_1 \geq t$. By the claim (v) we have

$$C(T_1, S(T_1), K, T_2) \geq (S(T_1) - K)_+ = C(T_1, S(T_1), K, T_1),$$

i.e., $V_{\mathcal{A}}(T_1) \geq 0$, for $t < T_1$. Hence $V_{\mathcal{A}}(t) \geq 0$, i.e., $C(t, S(t), K, T_2) \geq C(t, S(t), K, T_1)$, which is the claim.

(vii) We prove the statement for call options, the argument for put options being the same. Let $K_0, K_1 > 0$ and $0 < \theta < 1$ be given. Consider a portfolio $\mathcal{A}$ which is short one share of a call with strike $\theta K_1 + (1-\theta)K_0$ and maturity $T$, long $\theta$ shares of a call with strike $K_1$ and maturity $T$, long $(1-\theta)$ shares of a call with strike $K_0$ and maturity $T$. The value of this portfolio at maturity is

$$V_{\mathcal{A}}(T) = -(S(T) - (\theta K_1 + (1-\theta)K_0))_+ + \theta(S(T) - K_1)_+ + (1-\theta)(S(T) - K_0)_+.$$

The convexity of the function $f(x) = (S(T) - x)_+$ gives $\mathcal{V}_A(T) \geq 0$ and so $V_{\mathcal{A}}(t) \geq 0$ by the dominance principle. The latter inequality is

$$C(t, S(t), \theta K_1 + (1-\theta)K_0, T) \leq \theta C(t, S(t), K_1, T) + (1-\theta)C(t, S(t), K_0, T),$$

which is the claim for call options.

□

**Exercise 1.5.** *Consider the following table of European options prices at time $t = 0$:*

| | CALL | |
|---|---|---|
| **Maturity** | **Strike** | **Price** |
| *1 month* | *104* | *20* |
| *1 month* | *110* | *16* |
| *1 month* | *116* | *10* |
| *3 month* | *110* | *15* |

| | PUT | |
|---|---|---|
| **Maturity** | **Strike** | **Price** |
| *1 month* | *96* | *14* |
| *1 month* | *100* | *16* |
| *1 month* | *104* | *18* |
| *3 month* | *110* | *18* |

*Assume that the money market has interest rate $r = 0$ and that the price of the underlying asset at time $t = 0$ is $S(0) = 100$. Explain why these prices are incompatible with the dominance principle. Find a constant portfolio position which violates the dominance principle. HINT: Look for violations of the properties (iv)–(vii).*

**Exercise 1.6.** *Assume that the dominance principle holds and prove the following.*

*(viii)* *If $K_0 \leq K_1$, then $C(t, S(t), K_0, T) \geq C(t, S(t), K_1, T)$, i.e., the price of European call options is non-increasing with the strike price. Similarly the price of put options is non-decreasing with the strike price.*

*(ix)* $C(t, S(t), K, T) \leq S(t)$ *and* $P(t, S(t), K, T) \leq K e^{-r(T-t)}$.

**Exercise 1.7 (•).** *Assume that the dominance principle holds. Consider the European derivative $\mathcal{U}$ with maturity time $T$ and pay-off $Y$ given by $Y = \min[(S(T) - K_1)_+, (K_2 - S(T))_+]$, where $K_2 > K_1$ and $(x)_+ = max(0, x)$. Draw the pay-off function of the derivative. Find a constant portfolio consisting of European calls and puts expiring at time $T$ which replicates the value of $\mathcal{U}$ (i.e., whose value at any time $t < T$ equals the value of $\mathcal{U}$).*

**Exercise 1.8.** *Suppose $K, \Delta K > 0$. A butterfly spread on call options pays the amount*

$$(S(T) - K + \Delta K)_+ - 2(S(T) - K)_+ + (S(T) - K - \Delta K)_+$$

*at the maturity $T$. Draw the pay-off function of the derivative. Show that the value of this option is non-negative at any point of time.*

**Exercise 1.9 (•).** *The price of a contract at time $t$ is $N$ units of currency and it pays at the maturity date $T > t$ the amount $N + \alpha N(S(T) - K)_+$. Show that*

$$\alpha = \frac{1 - e^{-r(T-t)}}{C(t, S(t), K, T)} \tag{1.11}$$

*if $C(t, S(t), K, T) > 0$ and $N \neq 0$.*

## Optimal exercise time of American options

Consider now a no-dummy investor owning an American put option. When should the investor exercise the option? At any time $t < T$ we have, by (iii),

$$\text{either} \quad \widehat{P}(t, S(t), K, T) > (K - S(t))_+ \quad \text{or} \quad \widehat{P}(t, S(t), K, T) = (K - S(t))_+.$$

Exercising the American put at a time $t$ when the strict inequality $\widehat{P}(t, S(t), K, T) > (K - S(t))_+$ holds is a dummy decision, because the resulting pay-off is lower than the value of the derivative[11]. On the other hand, if the equality $\widehat{P}(t,S(t),K,T) = (K - S(t))_+$ holds at time $t$, then the optimal strategy for the investor is to exercise the American put, as in this case the pay-off equals the value of the derivative, i.e., the investor takes full advantage of the American put. This leads us to introduce the following definition.

**Definition 1.1.** *A time $t < T$ is called an* **optimal exercise time** *for the American put with value $\widehat{P}(t,S(t),K,T)$ if*

$$\widehat{P}(t,S(t),K,T) = (K - S(t))_+.$$

A similar definition can be justified for American call options, i.e., the optimal exercise time of the American call is a time $t$ at which $\widehat{C}(t,S(t),K,T) = (S(t) - K)_+$. However, assuming that the dominance principle holds (and that the money market has positive interest rate), we have $\widehat{C}(t,S(t),K,T) \geq C(t,S(t),K,T) > (S(t)-K)_+$, for $t < T$, see (v) in Theorem 1.1. It follows that, in an arbitrage-free market, *it is never optimal to exercise American call options prior to maturity when the underlying stock pays no dividend.* As opposed to this, it will be shown in Chapter 6 that when the underlying stock pays a dividend prior to the expiration date of the American call, it is optimal to exercise the American call just before the dividend is paid, provided the price of the stock is sufficiently high, see Theorem 6.9.

As in the absence of dividends the optimal strategy is to hold the American call until maturity, the dominance principle leads us to the following, last property on the fair price of options:

(x) When the underlying stock pays no dividend (and the money market has positive interest rate), the fair price of European calls and American call with equal parameters are the same, i.e.,

$$\widehat{C}(t,S(t),K,T) = C(t,S(t),K,T).$$

**Final remarks:** The properties (i)-(x) are quite well represented in real markets, thereby giving indirect support to the arbitrage-free principle. Note also that these properties depend only on the validity of the arbitrage-free principle (in the form of the dominance principle) and *not* on the specific market dynamics. In the following chapters we shall give an alternative proof of (some of) these properties by using explicit models for the price of stocks and options in the market.

**Exercise 1.10 (Comparison with market data).** *Call and put options on Nasdaq 100 are of European style and so they can be used to test the properties derived in this section. The market price of call and put options on Nasdaq 100, for different strikes and maturities, can be found at the homepage* `http://www.marketwatch.com/investing/index/ndx/options`. *Compile a table of prices for options "nearly at the money", for instance using the first 10 options in and out of the money (if an option appears with zero price it means*

[11] Of course the buyer might want to close the position on the American put for other reasons. In this case however it is more profitable to sell the option rather than exercising it.

*that it has not yet been traded; you can just skip it). Use these data to plot (with Matlab for instance) the price of call and put options in terms of the strike price and the time of maturity. Are the properties (vi), (viii), (viii) verified? Next use the put-call parity identity to compute the value of the interest rate $r$ for each pair of call-put with the same strike and maturity. What can you conclude? Do the data support the put-call parity?*

# Chapter 2

# Binomial markets

In this and the following two chapters we present a time-discrete model for the fair price of options first proposed in [4] and which is known under the name of **binomial options pricing model**. The model is very popular among practitioners due to its implementation simplicity. The present chapter deals with the dynamics of the underlying asset, which we assume to be a stock. The following two chapters are concerned with European and American derivatives on the stock. The time-continuum analogue of the binomial model is the Black-Scholes model, which will be studied in Chapter 6.

## 2.1 The binomial stock price

The **binomial asset price** is a model for the evolution in time of the price of financial assets. It is often applied to stocks, hence we denote by $S(t)$ the price of the asset at time $t$. We are interested in monitoring the stock price in some finite time interval $[0, T]$, where $T > 0$ could be for instance the expiration date of an option on the stock. The price of the stock at time $t = 0$ is denoted by $S(0)$ or $S_0$ and is assumed to be known.

The binomial stock price can only change at some given pre-defined times $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$; moreover the price at time $t_{i+1}$ depends only on the price at time $t_i$ and the result of "tossing a coin". Precisely, letting $u, d \in \mathbb{R}$, $u > d$, and $p \in (0, 1)$, we assume

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, & \text{with probability } p, \\ S(t_{i-1})e^d, & \text{with probability } 1-p, \end{cases}$$

for all $i = 1, \ldots, N$. Here we may interpret $p$ as the probability to get a head in a coin toss ($p = 1/2$ for a **fair coin**). We restrict to the standard binomial model, which assumes that the market parameters $u, d, p$ are time-independent and that the stock pays no dividend in the interval $[0, T]$. In the applications one typically chooses $u > 0$ and $d < 0$ (e.g., $d = -u$ is quite common), hence $u$ stands for "up", since $S(t_i) = S(t_{i-1})e^u > S(t_{i-1})$, while $d$ stands for "down", for $S(t_i) = S(t_{i-1})e^d < S(t_{i-1})$. In the first case we say that the stock price goes up at time $t_i$, in the second case that it goes down at time $t_i$.

Next we introduce a number of assumptions which simplify the analysis of the model without compromising its generality[1]. Firstly we assume that the times $t_0, t_1, \dots t_N$ are equidistant, that is

$$t_i - t_{i-1} = h > 0, \quad \text{for all } i = 1, \dots, n.$$

In the applications the value of $h$ must be chosen much smaller than $T$. Without loss of generality we can pick $h = 1$, and so

$$t_1 = 1, \quad t_2 = 2, \quad \dots, t_N = T = N,$$

with $N >> 1$. For instance, if $N = 67$ (the number of trading days in a period of 3 months), then $h = 1$ day and $S(t)$, for $t \in \{1, \dots N\}$, may refer to the closing price of the stock at each day. It is convenient to denote

$$\mathcal{I} = \{1, \dots, N\}.$$

Hence, from now on, we assume that the binomial stock price is determined by the rule $S(0) = S_0$ and

$$S(t) = \begin{cases} S(t-1)e^u, & \text{with probability } p \\ S(t-1)e^d, & \text{with probability } 1-p \end{cases}, \quad t \in \mathcal{I}. \tag{2.1}$$

**Remark 2.1** (Notation)**.** The notation used in the present notes is the same as in [3], although it is slightly different from the one used in the standard literature on the binomial model, see e.g., [6]. In fact the binomial stock price is more commonly written as

$$S(t) = \begin{cases} S(t-1)u, & \text{with probability } p \\ S(t-1)d, & \text{with probability } 1-p \end{cases},$$

with $0 < d < u$. All the results in the present text can be translated into the standard notation by the substitutions $e^u \to u$, $e^d \to d$. In our notation the log-returns of the stock take a slightly simpler form, which is useful when passing to the time-continuum limit (see Section 6.1).

Each possible sequence $(S(1), \dots, S(N))$ of the future stock prices determined by the binomial model is called a **path** of the stock price. Clearly, there exists $2^N$ possible paths of the stock price in a $N$-period model. Letting[2]

$$\{u, d\}^N = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_t = u \text{ or } x_t = d, \ t \in \mathcal{I}\}$$

[1]We come back to the general model in Section 2.4, where it is implemented with Matlab.

[2]Note carefully that in the set $\{u, d\}^N$, the letters $u, d$ mean "up" and "down" and should not be confused with the numerical parameters $u, d$ in the binomial stock price (2.1).

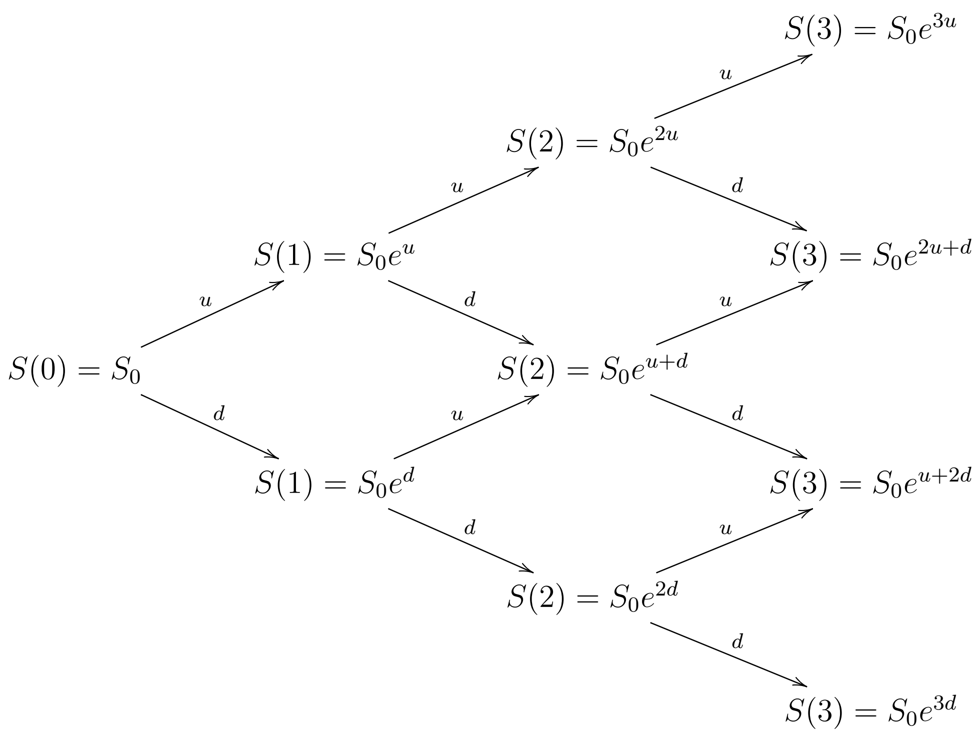
be the space of all possible $N$-sequences of "ups" and "downs", we obtain a unique path of the stock price $(S(1), \ldots, S(N))$ for each $x \in \{u, d\}^N$. For instance, for $N = 3$ and $x = (u, u, u)$ the corresponding stock price path is given by

$$S_0 \to S(1) = S_0 e^u \to S(2) = S(1)e^u = S_0 e^{2u} \to S(3) = S(2)e^u = S_0 e^{3u},$$

while for $x = (u, d, u)$ we obtain the path

$$S_0 \to S(1) = S_0 e^u \to S(2) = S(1)e^d = S_0 e^{u+d} \to S(3) = S(2)e^u = S_0 e^{2u+d}.$$

In general the possible paths of the stock price for the 3-period model can be represented as




which is an example of **binomial tree**. The admissible values for the binomial stock price $S(t)$ at time $t$ are given by

$$S(t) \in \{S_0 e^{ku+(t-k)d},\ k = 0, \ldots, t\},$$

for all $t \in \mathcal{I}$, where $k$ is the number of times that the price goes up up to the time $t$ included.

**Definition 2.1.** *Given $x = (x_1, \ldots x_N) \in \{u, d\}^N$, the binomial stock price $S(t, x)$ at time $t \in \mathcal{I}$ corresponding to $x$ is given by*

$$S(t, x) = S_0 \exp(x_1 + x_2 + \ldots x_t).$$

*The vector $S^x = (S(1,x), \dots S(N,x))$ is called the $x$-path of the binomial stock price. Moreover we define the probability*[3] *of the path $S^x$ as*

$$\mathbb{P}(S^x) = p^{N_u(x)}(1-p)^{N_d(x)},$$

*where $N_u(x)$ is the number of $u$'s in the sequence $x$ and $N_d(x) = N - N_u(x)$ is the number of $d$'s. The probability that the binomial stock price follows one of the two paths $S^x$, $S^y$ is given by $\mathbb{P}(S^x) + \mathbb{P}(S^y)$ (and similarly for any number of paths).*

**Theorem 2.1.**

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = 1.$$

*Proof.* The sum is

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = \sum_{x\in\{u,d\}^N} p^{N_u(x)}(1-p)^{N_d(x)} = (1-p)^N \sum_{x\in\{u,d\}^N} \left(\frac{p}{1-p}\right)^{N_u(x)},$$

where for the last equality we used $N_d(x) = N - N_u(x)$. Now, even though the sum is over $x \in \{u,d\}^N$, the terms to be summed depend only on $N_u(x)$. Hence, the paths $x$ with the same value of $N_u(x)$ (i.e., the same number of "ups") give the same contribution to the sum. It is left as an exercise to show that given $k \in \{0,1,\dots,N\}$, the number of paths $x$ for which $N_u(x) = k$ is given by the **binomial coefficient**:

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}.$$

Each of the term in the sum for which $N_u(x) = k$ equals $(p/(1-p))^k$, hence, as there are $\binom{N}{k}$ of them, we obtain

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = (1-p)^N \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p}{1-p}\right)^k.$$

The **binomial theorem** asserts that $(a+b)^N = \sum_{k=0}^N \binom{N}{k} a^k b^{N-k}$. Letting $a = p/(1-p)$ and $b = 1$ we conclude

$$\sum_{x\in\{u,d\}^N} \mathbb{P}(S^x) = (1-p)^N \left(1 + \frac{p}{1-p}\right)^N = 1.$$

□

[3] We shall say more about the probabilistic interpretation of the binomial model in Chapter 5.

## 2.2 Binomial markets

A 1+1 dimensional **binomial market** is a market that consists of a risky asset, say a stock, and a risk-free asset, such that the price of the risky asset is given by the binomial model (2.1) and the value of the risk-free asset at time $t \in \mathcal{I}$ is given by

$$B(t) = B_0 e^{rt}, \quad t \in \mathcal{I}, \tag{2.2}$$

where $B_0 = B(0)$ is the present (at time $t = 0$) value of the asset. Recall that we assume that the interest rate $r$ of the money market is constant (not necessarily positive). As we shall only consider markets with one stock and one risk-free asset, we refer to 1+1 dimensional binomial markets simply as binomial markets. The constants $u, d, r, p$ are called **market parameters**.

**Remark 2.2** (Notation)**.** As a follow-up to Remark 2.1, we mention that our notation for the value of the risk-free asset is also slightly different from the standard one. In fact, most of the literature on the binomial model, e.g. [6], denotes the value of the risk-free asset at time $t$ by $B(t) = B_0(1 + r)^t$. To translate our results into the standard notation one just needs to replace $e^r$ with $(1 + r)$. As already reported in Remark 2.1, we follow the notation used in [3].

**Remark 2.3** (Discounted price)**.** The **discounted price** of the stock in a binomial market is defined by $S^*(t) = e^{-rt}S(t)$ and has the following meaning: $S^*(t)$ is the amount that should be invested on the risk-free asset at time $t = 0$ in order that the value at time $t$ of this investment replicates the value of the stock at time $t$. Note that, whenever $r > 0$, i.e., as long as buying the risk-free asset ensures a positive return, we have $S^*(t) < S(t)$. The discounted price of the stock measures the loss in the stock value due to the "time-devaluation" of money expressed by the ratio $B_0/B(t) = e^{-rt}$. Put it differently, 1 dollar today ($t = 0$) has the same (financial) purchasing power of $e^{rt}$ dollars at time $t$ in the future. Hence $S^*(t) = S(t)/e^{rt}$ measures the future value of the stock relative to the present value of money.

**Definition 2.2.** *A* **portfolio process** *invested in a binomial market is a finite sequence*

$$\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}},$$

*where*[4] $h_S(t) \in \mathbb{R}$ *is the number of shares invested in the stock and* $h_B(t) \in \mathbb{R}$ *is the number of shares invested in the risk-free asset during the time interval* $(t-1, t]$, $t \in \mathcal{I}$*. The value of the portfolio process at time t is given by*

$$V(t) = h_S(t)S(t) + h_B(t)B(t). \tag{2.3}$$

[4] Recall that we assume that it is possible to trade any fractional number of shares, hence $h_S(t), h_B(t)$ can be any real number. Of course, in the applications they have to be rounded to integer numbers.

The initial position of the investor is given by $(h_S(0), h_B(0))$. Without loss of generality we assume that

$$h_S(0) = h_S(1), \quad h_B(0) = h_B(1), \tag{2.4}$$

i.e, $(h_S(1), h_B(1))$ is the investor position on the closed interval $[0, 1]$ (and not just in the semi-open interval $(0, 1]$). We recall that $h_S(t) > 0$ means that the investor has a long position on the stock in the interval $(t-1, t]$, while $h_S(t) < 0$ corresponds to a short position.

It is clear that the investor will change the position on the stock and the risk-free asset according to the path followed by the stock price, and so $(h_S(t), h_B(t))$ is in general path-dependent. When we want to emphasize the dependence of a portfolio position on the path of the stock price we shall write

$$h_S(t) = h_S(t, x), \quad h_B(t) = h_B(t, x).$$

Similarly, we write $V(t) = V(t, x)$ if we want to emphasize the dependence of the portfolio value on the path of the stock price. Clearly,

$$V(t, x) = h_S(t, x)S(t, x) + h_B(t, x)B(t).$$

We say that the portfolio process is self-financing if purchasing more shares of one asset is possible only by selling shares of the other asset for an equivalent value (and not by infusing new cash into the portfolio), and, conversely, if any cash obtained by selling one asset is immediately re-invested to buy shares of the other asset (and not withdrawn from the portfolio). To translate this condition into a mathematical formula, recall that $(h_S(t), h_B(t))$ is the investor position on the stock and the risk-free asset during the time interval $(t-1, t]$. Assume that the investor sells/buys shares of the assets at time $t$. Let $(h_S(t+1), h_B(t+1))$ be the new position on the stock and the risk-free asset in the interval $(t, t+1]$. The value of the portfolio process just before and immediately after changing the position at time $t$ is given respectively by

$$V(t) = h_S(t)S(t) + h_B(t)B(t), \quad V'(t) = h_S(t+1)S(t) + h_B(t+1)B(t).$$

The difference $V'(t) - V(t)$, if not zero, corresponds to cash withdrawn or added to the portfolio as a result of the change in the position on the assets. In a self-financing portfolio, however, this difference must be zero. We thus must have $V'(t) = V(t)$, which leads to the following definition.

**Definition 2.3.** *A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ invested in a binomial market is said to be* **self-financing** *if*

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) \tag{2.5}$$

*holds for all $t \in \mathcal{I}$.*

**Remark 2.4.** Constant (i.e., time-independent) portfolio processes are clearly self-financing. In fact, (2.5) holds if $h_S(t) = h_S(t-1)$ and $h_B(t) = h_B(t-1)$, for all $t \in \mathcal{I}$.

**Exercise 2.1.** *Denote $\Delta f = f(t) - f(t-1)$, for any function $f$ of $t \in \mathcal{I}$. Show that for any self-financing portfolio there holds*

$$\Delta V(t) = h_S(t)\Delta S(t) + h_B(t)\Delta B(t).$$

*Interpret the result.*

Our next purpose is to show that the value of a self-financing portfolio at time $t = N$ determines uniquely the value of the portfolio at any earlier time $t = 0, \dots, N-1$. This result is crucial to justify our definition of binomial fair price of European derivatives in the next chapter. We begin by fixing some notation. First we define the parameters $q_u, q_d$ as

$$q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = 1 - q_u = \frac{e^u - e^r}{e^u - e^d}. \tag{2.6}$$

Note the $(q_u, q_d)$ is the unique solution of the linear system

$$q_u + q_d = 1, \quad q_u e^u + q_d e^d = e^r. \tag{2.7}$$

Now, given a self-financing portfolio process, we denote

$$V^u(t) = h_S(t)S(t-1)e^u + h_B(t)B(t-1)e^r,$$

which is the value of the portfolio at time $t$ assuming that the stock price goes up at time $t$, and

$$V^d(t) = h_S(t)S(t-1)e^d + h_B(t)B(t-1)e^r$$

which is the value of the portfolio at time $t$, assuming that the stock price goes down at time $t$. We can now prove the main result of this section.

**Theorem 2.2.** *Let $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ be a self-financing portfolio process with value $V(N)$ at time $t = N$. The portfolio value $V(t)$ at earlier times satisfies the following recurrence formula:*

$$V(t) = e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)], \quad \text{for } t = 0, \dots, N-1. \tag{2.8}$$

*Moreover*

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots,x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x), \quad \text{for } t = 0, \dots, N-1. \tag{2.9}$$

*In particular, at time $t = 0$ we have*

$$V(0) = e^{-rN} \sum_{x\in\{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N, x). \tag{2.10}$$

*Proof.* Using the definition of $V^u(t), V^d(t)$, the right hand side of (2.8) is

$$\begin{aligned} e^{-r}[q_u V^u(t+1) + q_d V^d(t+1)] &= e^{-r}[q_u(h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r) \\ &\quad + q_d(h_S(t+1)S(t)e^d + h_B(t+1)B(t)e^r)] \\ &= e^{-r}[h_S(t+1)S(t)(q_u e^u + q_d e^d) + h_B(t+1)B(t)e^r(q_u + q_d)] \\ &= h_S(t+1)S(t) + h_B(t+1)B(t), \end{aligned} \tag{2.11}$$

where in the last step we used (2.7). By definition of self-financing portfolio, the last member in (2.11) equals $V(t)$, and so (2.8) is proved.

It remains to establish (2.9); we argue by induction on $t = 0, \dots, N-1$.

**Step 1.** We first prove (2.9) for $t = N-1$. In this case the sum in the right hand side of (2.9) is over two terms, one for which $x_N = u$ and one for which $x_N = d$. Hence we have to prove

$$V(N-1) = e^{-r}[q_u V(N, (x_1, \dots, x_{N-1}, u)) + q_d V(N, (x_1, \dots, x_{N-1}, d))]. \tag{2.12}$$

In the right hand side of (2.12) we recognize

$$V(N, (x_1, \dots, x_{N-1}, u)) = h_S(N)S(N-1)e^u + h_B(N)B(N-1)e^r = V^u(N)$$

and

$$V(N, (x_1, \dots, x_{N-1}, d)) = h_S(N)S(N-1)e^d + h_B(N)B(N-1)e^r = V^d(N),$$

and so (2.12) is equivalent to (2.8) at time $t = N-1$. As (2.8) has already been established for all times, the proof of (2.12) is completed.

**Step 2.** Now assume that (2.9) is true at time $t+1$, i.e.,

$$V(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, \dots x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x).$$

**Step 3.** We now prove (2.9) at time $t$. By the induction hypothesis of step 2 we have

$$V^u(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, \dots, x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, u, x_{t+2}, \dots, x_N),$$

$$V^d(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}, \dots, x_N) \in \{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, d, x_{t+2}, \dots, x_N).$$

Replacing in (2.8) we obtain

$$
\begin{aligned}
V(t) = e^{-r(N-t)} \Bigg[ q_u & \sum_{(x_{t+2},\dots,x_N)\in\{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, u, x_{t+2}, \dots, x_N) \\
& + q_d \sum_{(x_{t+2},\dots,x_N)\in\{u,d\}^{N-t-1}} q_{x_{t+2}} \cdots q_{x_N} V(N, x_1, \dots, x_t, d, x_{t+2}, \dots, x_N) \Bigg] \\
= e^{-r(N-t)} & \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x),
\end{aligned}
$$

which completes the proof. □

**Remark 2.5.** As the sum in (2.9) is over $(x_{t+1}, \dots, x_N)$ and $V(N)$ depends on the full path $x \in \{u, d\}^N$, then $V(t)$ depends only on the first $t$ steps of the path of the stock price, i.e.,

$$V(t) = V(t, x_1, x_2, \dots, x_t).$$

**Example.** We conclude this section with an example of application of (2.10). Assume $N = 2$ and consider a binomial market with the following parameters:

$$e^d = 2, \quad e^u = 3, \quad e^r = 1.$$

The parameters (2.6) are given by

$$q_u = -1, \quad q_d = 2.$$

The set of possible paths is

$$\{u, d\}^2 = \{(u, u), (u, d), (d, u), (d, d)\}.$$

Hence the sum in (2.10) is over 4 terms. As $e^{-rN} = 1$ in this example, we have

$$
\begin{aligned}
V(0) &= q_u^2 V(2, (u, u)) + q_u q_d V(2, (u, d)) + q_u q_d V(2, (d, u)) + q_d^2 V(2, (d, d)) \\
&= V(2, (u, u)) - 2V(2, (u, d)) - 2V(2, (d, u)) + 4V(2, (d, d)).
\end{aligned}
$$

Assuming for instance that the possible portfolio values at time $t = 2$ are

$$V(2, (u, u)) = 1, \quad V(2, (u, d)) = 2, \quad V(2, (d, u)) = -1, \quad V(2, (d, d)) = 4,$$

we obtain that the portfolio value at time $t = 0$ is $V(0) = 15$.

**Exercise 2.2** ($\star$)**.** *Use the recurrence formula* (2.8) *to compute all possible values of* $V(t)$ *for* $t = 0, 1$ *in the example above.*

**Exercise 2.3** ($\star$)**.** *Assume* $S_0 = B_0 = 1$ *in the example above and find a self-financing portfolio process with value* $V(t)$.

## 2.3 Arbitrage portfolio

Our next purpose is to prove that the binomial market introduced in the previous section is arbitrage free, provided its parameters satisfy $d < r < u$ (no condition is required on the probability $p$). To achieve this we first need to introduce the precise definition of arbitrage portfolio.

**Definition 2.4.** *A portfolio process* $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ *invested in a binomial market is called an* **arbitrage portfolio process** *if its value* $V(t)$ *satisfies*

*1)* $V(0) = 0$*;*

*2)* $V(N, x) \geq 0$*, for all* $x \in \{u, d\}^N$*;*

*3) There exists* $y \in \{u, d\}^N$ *such that* $V(N, y) > 0$*.*

*We say that the binomial market is* **arbitrage-free** *if there exists no self-financing arbitrage portfolio process invested in the stock and the risk-free asset.*

Let us comment on the previous definition. Condition 1) means that no initial wealth is required to set up the portfolio, i.e., the long and short positions on the two assets are perfectly balanced. In particular, anyone can (in principle) open such portfolio (no initial wealth is required). Condition 2) means that the investor is sure not to loose money with this investment: regardless of the path followed by the stock price, the return of the portfolio is always non-negative. Condition 3) means that there is a strictly positive probability to make a profit, since along at least one path of the stock price the return of the portfolio is strictly positive.

**Theorem 2.3.** *The binomial market is arbitrage free if and only if* $d < r < u$*.*

*Proof.* We divide the proof in two steps. In step 1 we prove the claim for the **1-period model**, i.e., $N = 1$. The generalization to the **multiperiod model** ($N > 1$) is carried out in step 2. Note also the claim of the theorem is logically equivalent to the following: *There exists a self-financing arbitrage portfolio in the binomial market if and only if* $r \notin (d, u)$. It is the latter claim which is actually proved below.

**Step 1: the 1-period model**. Because of our convention (2.4), we can set

$$h_S(0) = h_S(1) = h_S, \quad h_B(0) = h_B(1) = h_B,$$

i.e., the portfolio position in the 1-period model is constant (and thus self-financing) over the interval $[0, 1]$. The value of the portfolio at time $t = 0$ is

$$V(0) = h_S S_0 + h_B B_0,$$

while at time $t = 1$ it is given by one of the following:

$$V(1) = V(1, u) = h_S S_0 e^u + h_B B_0 e^r,$$

if the stock price goes up at time $t = 1$, or

$$V(1) = V(1, d) = h_S S_0 e^d + h_B B_0 e^r,$$

if the stock price goes down at time $t = 1$. Thus the portfolio is an arbitrage if $V(0) = 0$, i.e.,

$$h_S S_0 + h_B B_0 = 0, \tag{2.13}$$

if $V(1) \geq 0$, i.e.,

$$h_S S_0 e^u + h_B B_0 e^r \geq 0, \tag{2.14}$$

$$h_S S_0 e^d + h_B B_0 e^r \geq 0, \tag{2.15}$$

and if at least one of the inequalities in (2.14)-(2.15) is strict. Now assume that $(h_S, h_B)$ is an arbitrage portfolio. From (2.13) we have $h_B B_0 = -h_S S_0$ and therefore (2.14)-(2.15) become

$$h_S S_0 (e^u - e^r) \geq 0, \tag{2.16}$$

$$h_S S_0 (e^d - e^r) \geq 0. \tag{2.17}$$

Since at least one of the inequalities must be strict, then $h_S \neq 0$. If $h_S > 0$, then (2.16) gives $u \geq r$, while (2.17) gives $d \geq r$. As $u > d$, the last two inequalities are equivalent to $r \leq d$. Similarly, for $h_S < 0$ we obtain $u \leq r$ and $d \leq r$ which, again due to $u > d$, are equivalent to $r \geq u$. We conclude that the existence of an arbitrage portfolio implies $r \leq d$ or $r \geq u$, that is $r \notin (d, u)$. This proves that for $r \in (d, u)$ there is no arbitrage portfolio in the one period model, and thus the "if" part of the theorem is proved for $N = 1$. To prove the "only if" part, i.e., the fact that $r \in (d, u)$ is necessary for the absence of arbitrages, we construct an arbitrage portfolio when $r \notin (d, u)$. Assume $r \leq d$. Let us pick $h_S = 1$ and $h_B = -S_0/B_0$. Then $V(0) = 0$. Moreover (2.15) is trivially satisfied and, since $u > d$,

$$h_S S_0 e^u + h_B B_0 e^r = S_0(e^u - e^r) > S_0(e^d - e^r) \geq 0,$$

hence the inequality (2.14) is strict. This shows that one can construct an arbitrage portfolio if $r \leq d$ and a similar argument can be used to find an arbitrage portfolio when $r \geq u$. The proof of the theorem for the 1-period model is complete.

**Step 2: the multiperiod model.** Let $r \notin (d, u)$. As shown in the previous step, there exists an arbitrage portfolio $(h_S, h_B)$ in the single period model. We can now build a self-financing arbitrage portfolio process $\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}}$ for the multiperiod model by investing at time $t = 1$ the whole value of the portfolio $(h_S, h_B)$ in the risk-free asset. The value of this portfolio process satisfies $V(0) = 0$ and $V(N, x) = V(1, x)e^{r(N-1)} \geq 0$ along every path $x \in \{u, d\}^N$. Moreover, since $(h_S, h_B)$ is an arbitrage, then $V(1, y) > 0$ along some path $y \in \{u, d\}^N$ and thus $V(N, y) > 0$. Hence the constructed self-financing portfolio process $\{(h_S(t), h_B(t))\}_{t \in \mathcal{I}}$ is an arbitrage and the "if" part of the theorem is proved. To prove the "only if" part for the multiperiod model, we use that, by Theorem 2.2,

$$V(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} V(N, x),$$

where $q_u$, $q_d$ are given by (2.6), $N_u(x)$ is the number of "ups" in $x$ and $N_d(x) = N - N_u(x)$ the number of "downs". Now, assume that the portfolio is an arbitrage. Then $V(0) = 0$ and $V(N, x) \geq 0$. Of course, the above sum can be restricted to the paths along which $V(N, x) > 0$, which exist since the portfolio is an arbitrage. But then the sum can be zero only if either one of the factors $q_u$, $q_d$ is zero, or if they have opposite sign. Since $u > d$, the denominator in the expressions (2.6) is positive, hence

$$q_u = 0, \quad \text{resp. } q_d = 0 \Rightarrow r = d, \quad \text{resp. } u = r,$$

$$(q_u > 0, q_d < 0), \quad \text{resp.} \quad (q_u < 0, q_d > 0) \Rightarrow u < r, \quad \text{resp. } r < d.$$

We conclude that the existence of a self-financing arbitrage portfolio entails $r \notin (d, u)$, which completes the proof of the theorem. □

**Remark 2.6.** The condition $d < r < u$ is equivalent to require that the parameters $q_u$, $q_d$ given by (2.6) satisfy $q_u \in (0, 1)$ and $q_d = 1 - q_u \in (0, 1)$, i.e., that the pair $(q_u, q_d)$ defines a probability, which is called **risk-neutral probability** or **martingale probability**; the reason for this terminology will become clear in Chapter 5.

**Exercise 2.4.** *Find an arbitrage portfolio in the binomial market given in the example at the end of Section 2.2.*

## 2.4 Computation of the binomial stock price with Matlab

In this section we learn how to compute the binomial stock price with Matlab. More precisely, our goal is to construct a binomial tree for the stock price in some interval $[0, T]$, with $T > 0$ measured in fraction of years. Let us start by dividing the interval $[0, T]$ into $N$ subintervals of length $h = T/N$, i.e.,

$$0 = t_1 < t_2 < \cdots < t_{N+1} = T, \quad t_{i+1} = t_i + h, \quad i = 1, \ldots, N.$$

Let $S(i) = S(t_i)$. We define the binomial stock price on the given partition of the interval $[0, T]$ as

$$S(i+1) = \begin{cases} S(i)e^u, & \text{with probability } p \\ S(i)e^d, & \text{with probability } 1-p \end{cases}, \quad i \in \mathcal{I}.$$

Now we assume that the smaller is $h$, the lower will be the change of the stock price on each subinterval, i.e., the smaller are the parameters $u, d$. More precisely, we assume that the parameters $u$ and $d$ can be written in the form

$$u = \alpha h + \sigma \sqrt{\frac{1-p}{p}} \sqrt{h}, \quad d = \alpha h - \sigma \sqrt{\frac{p}{1-p}} \sqrt{h}, \tag{2.18}$$

for some constant parameters $\sigma > 0$ and $\alpha \in \mathbb{R}$, which we call, respectively, the **instantaneous mean of log-return** and the **instantaneous volatility** of the binomial stock price in the interval $[0, T]$. The reason to choose the parameters $u, d$ in this form will become clear in Chapter 6, where we prove that this choice makes the binomial stock price converge (in distribution) to the geometric Brownian motion in the continuum limit $h \to 0$. Note also that, by inverting (2.18),

$$\alpha = \frac{1}{h}[pu + (1-p)d], \quad \sigma = \frac{u-d}{\sqrt{h}} \sqrt{p(1-p)}. \tag{2.19}$$

The parameter $\alpha$ measures the average movement of the binomial stock price in the interval $[0, T]$, while $\sigma$ measures the average amplitude of the oscillations of the binomial stock price. The quantity $\sigma^2$ is called **instantaneous variance**.

The following code defines the Matlab function *BinomialStock* which generates the binomial tree for the stock price on the partition $Q = \{t_1, \dots t_{N+1}\}$ of the interval $[0, T]$:

```
function [Q,S]=BinomialStock(p,alpha,sigma,s,T,N)
h=T/N;
u=alpha*h+sigma*sqrt(h)*sqrt((1-p)/p);
d=alpha*h-sigma*sqrt(h)*sqrt(p/(1-p));
Q=zeros(N+1,1);
S=zeros(N+1);
Q(1)=0;
S(1,1)=s;
for j=1:N
Q(j+1)=j*h;
S(1,j+1)=S(1,j)*exp(u);
for i=1:j
S(i+1,j+1)=S(i,j)*exp(d);
end
end
```

The arguments of the function are the parameters $p, \alpha, \sigma$, the time $T > 0$ (expressed in fraction of years), the initial price of the stock $s$ and the number of steps $N$ in the binomial model. The function returns a column vector $Q$ containing the times $t_1, \dots t_{N+1}$ of the partition, and an upper-triangular $(N+1) \times (N+1)$ matrix $S$. The column $j$ of $S$ contains, in decreasing order along rows, the possible prices of the stock at time $t_j = (j-1)h$. A path of the stock price is obtained by moving from each column to the next one by either staying in the same row (which means that the price went up at this step) or going down one row (which means that the price went down at this step). For example, by running the command

```
[Q,S]=BinomialStock(0.5,0.01,0.2,10,1/12,5);
```

we get the output

$$Q = \begin{matrix} 0 \\ 0.0167 \\ 0.0333 \\ 0.0500 \\ 0.0667 \\ 0.0833 \end{matrix}$$

$$S = \begin{matrix} 10.0000 & 10.2633 & 10.5335 & 10.8108 & 11.0954 & 11.3875 \\ 0 & 9.7467 & 10.0033 & 10.2667 & 10.5370 & 10.8144 \\ 0 & 0 & 9.4999 & 9.7500 & 10.0067 & 10.2701 \\ 0 & 0 & 0 & 9.2593 & 9.5030 & 9.7532 \\ 0 & 0 & 0 & 0 & 9.0248 & 9.2624 \\ 0 & 0 & 0 & 0 & 0 & 8.7962 \end{matrix} \tag{2.20}$$

As $T = 1/12$ years and $N = 5$, the matrix $S$ shows the stock prices in a period of 1 month with intervals of $1/(12 * 5)$ years, i.e., about 6 days[5]. The mean of log-return of the stock is $\alpha = 1\%$, while its volatility is $\sigma = 20\%$.

**Remark 2.7.** Throughout these notes, the computations in Matlab are performed in high precision arithmetic and the results are truncated to the fourth decimal digit. In the applications, where the results correspond to prices expressed in some unit of currency, a cruder truncation may be required.

**Exercise 2.5.** *Define a function RandomPath(S) that generates a random path from the binomial tree S created with the function BinomialStock and compute its probability. Plot the price of the stock as a function of time.*

Recall that in the applications to real-word problems the number $N$ should be chosen sufficiently large (i.e., $h$ should be small compared to $T$), which makes it practically impossible to generate all possible $2^N$ paths of the stock price. Even for a relatively small number of steps, like $N \approx 50$, the number of admissible paths is too big. Note also that the probability of each single path is practically zero for $N \gtrsim 20$. For instance, if we assume $p = 1/2$, then each single path has equal probability $\mathbb{P}(S^x) = (1/2)^N$, which is approximately $10^{-4}\%$ for $N = 20$. The following code defines a Matlab function *ProbStock(S,t,a,b,p)* which computes

[5] We do not adjust our calculations to take into account that markets are closed in the week-ends.

the (much more interesting) probability that the price $S(t)$ lies within a given interval $(a, b)$, i.e., $\mathbb{P}(a < S(t) < b)$.

```
function Prob=ProbStock(S,t,a,b,p)
ind=find(S(:,t+1)>a & S(:,t+1)<b);
numups=t+1-ind;
k=length(numups);
probpath=0;
for j= 1:k
probpath=probpath+nchoosek(t,numups(j))*p^numups(j)*(1-p)^(t-numups(j));
end
Prob=probpath;
```

The function computes the probability $\mathbb{P}(a < S(t) < b)$ using Definition 2.1, i.e., by summing up the probability of each path which leads to a price $S(t) \in (a, b)$. As $S_{ij}$ in the binomial tree $S$ is the price at time $j$ assuming that the price goes down $i - 1$ times, the number of ups at time $t = j$ is given by $j + 1 - i$. The variable *numups* in the code above contains therefore the number of "ups" for each price at time $t$ which lies in the interval $(a, b)$.

# Chapter 3

# European derivatives

This chapter deals with the pricing theory of European derivatives on a single stock under the assumption that the price of the underlying is given by $S(0) = S_0$ at time $t = 0$ and

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \quad t \in \mathcal{I} = \{1, \dots, N\}.$$

Here $0 < p < 1$ and $u > d$. $S(t)$ is the binomial stock price at time $t \in \mathcal{I}$. It is assumed that the initial price $S_0$ is known. We say that the price goes up at time $t$ if $S(t) = S(t-1)e^u$ and that it goes down at time $t$ if $S(t) = S(t-1)e^d$ (although this terminology is strictly correct only if $d < 0$ and $u > 0$, which is most often the case in the applications). Moreover we assume the existence of a risk-free asset with value

$$B(t) = B_0 e^{rt}, \quad t \in \mathcal{I},$$

where $B_0 = B(0)$ is the initial value of the risk-free asset and $r$ is the constant interest rate of the money market. We impose $d < r < u$. In particular, the market is arbitrage-free and the risk-neutral probability is well defined:

$$q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = \frac{e^u - e^r}{e^u - e^d}, \qquad q_u, q_d \in (0,1), \; q_u + q_d = 1.$$

We denote by $\{(h_S(t), h_B(t)\}_{t \in \mathcal{I}}$ a portfolio process invested in the stock and the risk-free asset, where $(h_S(t), h_B(t))$ is the portfolio position in the interval $(t-1, t]$ and $h_S(0) = h_S(1)$, $h_B(0) = h_B(1)$. The value at time $t$ of the portfolio is $V(t) = h_S(t)S(t) + h_B(t)B(t)$. Recall that the portfolio process is said to be self-financing if

$$V(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1),$$

which means that no cash is ever withdrawn or added to the portfolio. The value of a self-financing portfolio process at time $t$ is uniquely determined by its value at time $N$. In fact we have the formula

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1}, \dots x_N) \in \{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} V(N, x). \tag{3.1}$$

which we proved in Theorem 2.2.

**Definition 3.1.** *A portfolio process* $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ *is called* **predictable** *if there exist* $N$ *functions* $H_1, \dots H_N$ *such that* $H_t : (0,\infty)^t \to \mathbb{R}^2$ *and*

$$(h_S(t), h_B(t)) = H_t(S_0, \dots, S(t-1)), \quad t \in \mathcal{I}.$$

In particular, for a predictable portfolio process, the position $(h_S(t), h_B(t))$ is a **deterministic function** of the stock price up to time $t-1$ and thus is depends on the path $x \in \{u, d\}^N$ only through the stock prices:

$$(h_S(t, x), h_B(t, x)) = H_t(S_0, S(1, x), \dots, S(t-1, x)).$$

As $S(t, x) = S_0 \exp(x_1 + \cdots + x_t)$ is independent of the steps *after* time $t$, we conclude that the position of a predictable portfolio process in the interval $(t-1, t]$ is determined by the information available *at* time $t-1$, which is the reason for calling it "predictable".

## 3.1 The binomial price of European derivatives

Consider a European derivative on the stock expiring at time $T = N$. The derivative is called **standard** if its pay-off depends only on the price of the stock at maturity, i.e., $Y = g(S(N))$, for some function $g : (0,\infty) \to \mathbb{R}$, which is called the **pay-off function** of the derivative. The European call, for instance, is a standard European derivative with pay-off function $g(z) = (z - K)_+$, where $K > 0$ is the strike price of the call. In the case of **non-standard** European derivatives, the pay-off is a function of the stock price at all times earlier and including $N$, i.e., $Y = g(S(1), \dots, S(N))$. Of course, in both cases the pay-off depends on the path followed by the stock price, in particular,

$$Y(x) = g(S(N, x)), \quad \text{for standard European derivatives}$$

and

$$Y(x) = g(S(1, x), S(2, x), \dots, S(N, x)), \quad \text{for non-standard European derivatives.}$$

Now, assume that a European derivative is sold at time $t < T$ for the price $\Pi_Y(t)$. The first concern of the seller is to **hedge** the derivative, i.e., to invest the premium $\Pi_Y(t)$ in such a way that the seller portfolio value at the expiration date is enough to pay-off the buyer of the derivative. We assume that the seller invests the premium in the 1+1 dimensional binomial market consisting of the underlying stock and the risk-free asset (**delta-hedging**).

**Definition 3.2.** *An* **hedging** *portfolio process for the European derivative with pay-off* $Y$ *and maturity* $T = N$ *is a portfolio process* $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ *invested in the underlying stock and the risk-free asset such that its value* $V(t)$ *satisfies* $V(N) = Y$*; the latter equality must be satisfied for all possible paths of the price of the underlying stock, i.e.,* $V(N, x) = Y(x)$*, for all* $x \in \{u, d\}^N$.

It follows by (3.1) that the value $V(t)$ of *any* self-financing hedging portfolio at time $t$ is given by

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1, \dots, x_N). \tag{3.2}$$

**Definition 3.3.** *The* **binomial (fair) price** *of the European derivative with pay-off $Y$ and maturity $T = N$ is given by*

$$\Pi_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x_1, \dots, x_N), \tag{3.3}$$

*for $t = 0, \dots, N-1$, while $\Pi_Y(N) := Y$. In other words, the binomial price of the European derivative equals the value of self-financing portfolios hedging the derivative.*

**Remark 3.1.** The identification of the fair price of the derivative with the value of self-financing hedging portfolios can be motivated as follows. Firstly the only purpose of the hedging portfolio is to pay-off the buyer of the derivative; the seller does not try to ensure a profit from the derivative (as it would be if $V(N) > Y$). The second reason is the absence of cash flow. In fact, if the writer needs to add cash to the portfolio in order to hedge the derivative, then the contract is clearly unfair for the writer. Conversely, if the writer could withdraw cash from the portfolio and still be able to hedge the derivative, then the contract would be clearly unfair for the buyer[1].

**Remark 3.2.** Note carefully that we have *not* proved yet that hedging self-financing portfolios exist. However we know that, if they exist, their value at time $t$ is given by (3.2). The existence of self-financing hedging portfolios is proved in Theorem 3.3 below.

**Remark 3.3.** By definition, hedging portfolios of European derivatives are also **replicating** portfolios, i.e., the equality $V(t) = \Pi_Y(t)$ holds at all time (and not only at maturity). We shall see in Chapter 4 that this is not the case for American derivatives.

By Remark 2.5, the value $\Pi_Y(t)$ depends on the path of the stock price up to time $t$, i.e.,

$$\Pi_Y(t) = \Pi_Y(t, x_1, \dots, x_t).$$

Hence the binomial price at time $t$ of European derivatives can be computed using the information available up to time $t$. For example, since by Definition 2.1 we have

$$S(N, x) = S_0 \exp(x_1 + \dots + x_N), \quad S(t, x) = S_0 \exp(x_1 + \dots + x_t)$$

then

$$S(N, x) = S(t) \exp(x_{t+1} + \dots + x_N),$$

[1]We take the opportunity to make the obvious remark that our theory treats the buyer and the seller on equal foot.

and therefore the binomial price for the standard European derivative with pay-off $Y = g(S(N))$ can be written as

$$\Pi_Y(t) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(S(t)\exp(x_{t+1} + \cdots + x_N)). \tag{3.4}$$

This shows that the binomial price at time $t$ of standard European derivatives is a deterministic function of $S(t)$, namely

$$\Pi_Y(t) = f_g(t, S(t)), \tag{3.5}$$

where

$$f_g(t, z) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} g(z\exp(x_{t+1} + \cdots + x_N)).$$

In the particular case of the European call (resp. put) option with strike $K$ and maturity $T = N$, the binomial fair price at time $t = 0, \dots, N-1$ can be written in the form $C(t, S(t), K, N)$ (resp. $P(t, S(t), K, N)$), where

$$C(t, S(t), K, N) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} (S(t)\exp(x_{t+1} + \cdots + x_N) - K)_+, \tag{3.6}$$

$$P(t, S(t), K, N) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} (K - S(t)\exp(x_{t+1} + \cdots + x_N))_+. \tag{3.7}$$

We use these formula to give an alternative proof of Theorem 1.2, which does not invoke the dominance principle.

**Theorem 3.1** (∗). *The binomial price of European calls and puts satisfy the properties in Theorem 1.2, that is*

*1 The put-call parity holds*

$$S(t) - C(t, S(t), K, N) = Ke^{-r(N-t)} - P(t, S(t), K, N).$$

*2 If $r \geq 0$, then $C(t, S(t), K, N) \geq (S(t) - K)_+$; the strict inequality $C(t, S(t), K, N) > (S(t) - K)_+$ holds when $r > 0$.*

*3 If $r \geq 0$, the map $N \to C(t, S(t), K, N)$ is non-decreasing.*

*4 The maps $K \to C(t, S(t), K, N)$ and $K \to P(t, S(t), K, N)$ are convex.*

*Proof.* Let $k$ be the number of $u$ in $(x_{t+1},\dots,x_N)$ and $N-t-k$ be the number of $d$. Then

$$e^{x_{t+1}+\dots+x_N} = e^{ku+(N-t-k)d}, \quad q_{x_{t+1}}\cdots q_{x_N} = q_u^k q_d^{N-t-k}.$$

Now, the number of paths $(x_{t+1},\dots,x_N)$ for which the number of $u$ is $k$ is given by the binomial coefficient $\binom{N-t}{k}$. Hence we can rewrite (3.6)-(3.7) as

$$C(t,S(t),K,N) = e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}(S(t)e^{ku+(N-t-k)d}-K)_+,$$

$$P(t,S(t),K,N) = e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}(K-S(t)e^{ku+(N-t-k)d})_+.$$

We can now prove the properties 1-4.

1 We have

$$C(t,S(t),K,N)-P(t,S(t),K,N) = e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}$$
$$\times\,[(S(t)e^{ku+(N-t-k)d}-K)_+ - (K-S(t)e^{ku+(N-t-k)d})_+].$$

Using that $(z-K)_+-(K-z)_+ = z-K$, for all $z\in\mathbb{R}$, we obtain

$$\begin{aligned}C-P &= e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}(S(t)e^{ku+(N-t-k)d}-K)\\ &= S(t)e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}e^{ku+(N-t-k)d}\\ &\quad - K\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k} = S(t)I_1 - KI_2.\end{aligned}$$

Hence the put-call parity follows if we show that $I_2 = e^{-r(N-t)}$ and $I_1 = 1$. We have

$$\begin{aligned}I_1 &= e^{-r(N-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}e^{ku+(N-t-k)d}\\ &= e^{-r(N-t)}(q_d e^d)^{N-t}\sum_{k=0}^{N-t}\binom{N-t}{k}\Big(\frac{q_u e^u}{q_d e^d}\Big)^k.\end{aligned}$$

Using the binomial theorem $(1+a)^N = \sum_{k=0}^N\binom{N}{k}a^k$ and the identity $q_u e^u + q_d e^d = e^r$ we obtain

$$\begin{aligned}I_1 &= e^{-r(N-t)}(q_d e^d)^{N-t}\Big(1+\frac{q_u e^u}{q_d e^d}\Big)^{N-t}\\ &= e^{-r(N-t)}(q_u e^u + q_d e^d)^{N-t} = 1.\end{aligned}$$

The proof that $I_2 = e^{-r(N-t)}$ is similar.

2 The proof follows by the put-call parity as in Theorem 1.2.

3 We want to show that

$$C(t,S(t),K,N) \le C(t,S(t),K,N+1).$$

The following proof is quite technical. In $C(N+1) := C(t,S(t),K,N+1)$ we replace the *Pascal identity*

$$\binom{N+1-t}{k} = \binom{N-t}{k-1} + \binom{N-t}{k}$$

(with the convention $\binom{N}{-1}=0$) and obtain

$$\begin{aligned} C(N+1) =& e^{-r(N+1-t)}\Bigg\{\sum_{k=1}^{N+1-t}\binom{N-t}{k-1}q_u^k q_d^{N+1-t-k}(S(t)e^{ku+(N+1-t-k)d}-K)_+ \\ &+\sum_{k=0}^{N+1-t}\binom{N-t}{k}q_u^k q_d^{N+1-t-k}(S(t)e^{ku+(N+1-t-k)d}-K)_+\Bigg\}. \end{aligned}$$

In the first sum we make the change of index $j=k-1$, while for the second sum we use that it is greater than the sum extended only up to $N-t$ (i.e., we neglect the last term $k=N+1-t$). So doing we obtain

$$\begin{aligned} C(N+1) \ge\ & e^{-r(N+1-t)}\Bigg\{\sum_{j=0}^{N-t}\binom{N-t}{j}q_u^{j+1}q_d^{N-t-j}(S(t)e^{(j+1)u+(N-t-j)d}-K)_+ \\ &+\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N+1-t-k}(S(t)e^{ku+(N+1-t-k)d}-K)_+\Bigg\} \\ =\ & e^{-r(N+1-t)}\Bigg\{\sum_{j=0}^{N-t}\binom{N-t}{j}q_u^j q_d^{N-t-j}q_u(S(t)e^{ju+(N-t-j)d}e^u-K)_+ \\ &+\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k}q_d(S(t)e^{ku+(N-t-k)d}e^d-K)_+\Bigg\} \\ =\ & e^{-r(N+1-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k} \\ &\times[(S(t)e^{ku+(N-t-k)d}q_u e^u - Kq_u)_+ + (S(t)e^{ku+(N-t-k)d}q_d e^d - Kq_d)_+]. \end{aligned}$$

Using the simple inequality $(y)_+ + (z)_+ \ge (y+z)_+$, we obtain

$$\begin{aligned} C(N+1) \ge\ & e^{-r(N+1-t)}\sum_{k=0}^{N-t}\binom{N-t}{k}q_u^k q_d^{N-t-k} \\ &\times[(S(t)e^{ku+(N-t-k)d}(q_u e^u + q_d e^d) - K(q_u+q_d))_+]. \end{aligned}$$

As $q_u e^u + q_d e^d = e^r$, $q_u + q_d = 1$ and $r \geq 0$ we find

$$\begin{aligned}
C(N+1) &\geq e^{-r(N+1-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d}e^r - K)_+ \\
&= e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d} - Ke^{-r})_+ \\
&\geq C(N).
\end{aligned}$$

4 The only dependence on $K$ of the functions $C(t, S(t), K, N), P(t, S(t), K, N)$ is through the terms $(z-K)_+$, $(K-z)_+$. As both these functions are convex in $K$ (draw a picture), the result follows.

□

**Exercise 3.1.** *Show that the binomial price at time $t$ of non-standard European derivatives is a deterministic function of $S(0), S(1), \dots, S(t)$.*

Now let $\Pi^u_Y(t)$ denote the binomial price of the European derivative at time $t$ assuming that the stock price goes up at time $t$ (i.e., $S(t) = S(t-1)e^u$, or equivalently, $x_t = u$), and similarly define $\Pi^d_Y(t)$, with "up" replaced by "down". By the proven formula (2.8) in Theorem 2.2 we have

**Theorem 3.2.** *The binomial price of European derivatives satisfies the recurrence formula*

$$\Pi_Y(N) = Y, \quad \textit{and} \quad \Pi_Y(t) = e^{-r}[q_u \Pi^u_Y(t+1) + q_d \Pi^d_Y(t+1)], \quad \textit{for } t \in \{0, \dots, N-1\}. \quad (3.8)$$

The recurrence formula (3.8) is very useful to compute the binomial price of standard European derivatives, as shown in the next example.

### 3.1.1 Example: A standard European derivative

Consider the standard European derivative with pay-off $Y = (\sqrt{S(2)} - 1)_+$ at maturity time $T = 2$. Assume that the market parameters are given by

$$u = \log 2, \quad d = 0, \quad r = \log(4/3), \quad p = 1/4.$$

Assume also $S_0 = 1$. Compute the possible paths of the derivative price $\Pi_Y(t)$. Compute also the probability that the derivative expires in the money.

**Solution:** The stock price and the risk-free asset satisfy

$$S(t) = \begin{cases} S(t-1)e^u \\ S(t-1)e^d \end{cases}, \quad B(t) = B_0 e^{rt} \qquad t \in \{1, 2\}$$

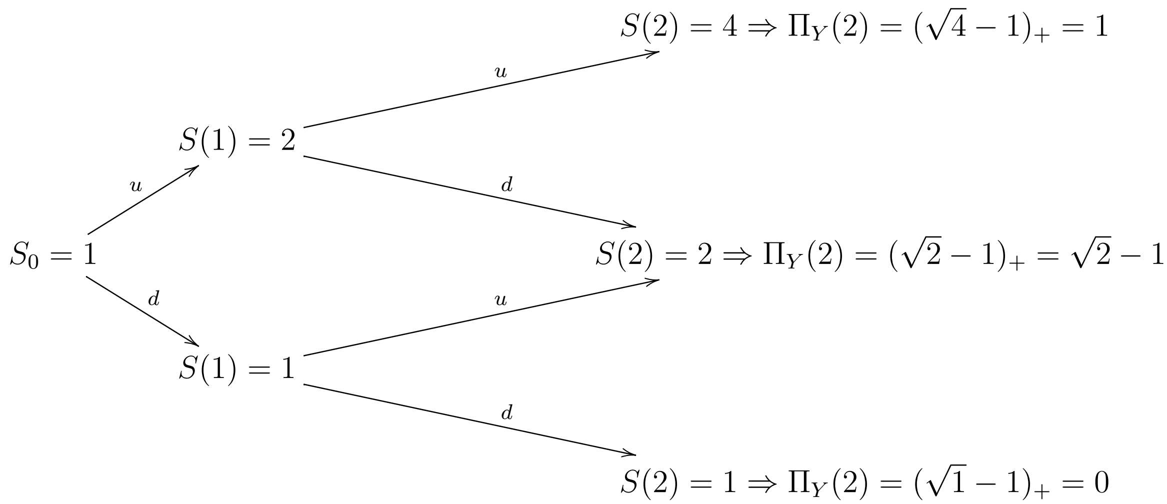
where

$$e^u = 2, \quad e^d = 1, \quad e^r = 4/3$$

Hence

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1}{3}, \quad q_d = 1 - q_u = \frac{2}{3}.$$

Now, let us write the binomial tree of the stock price, including the possible values of the derivative at the expiration time $T = 2$ (where we use that $\Pi_Y(2) = Y$):




Using the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1))$$

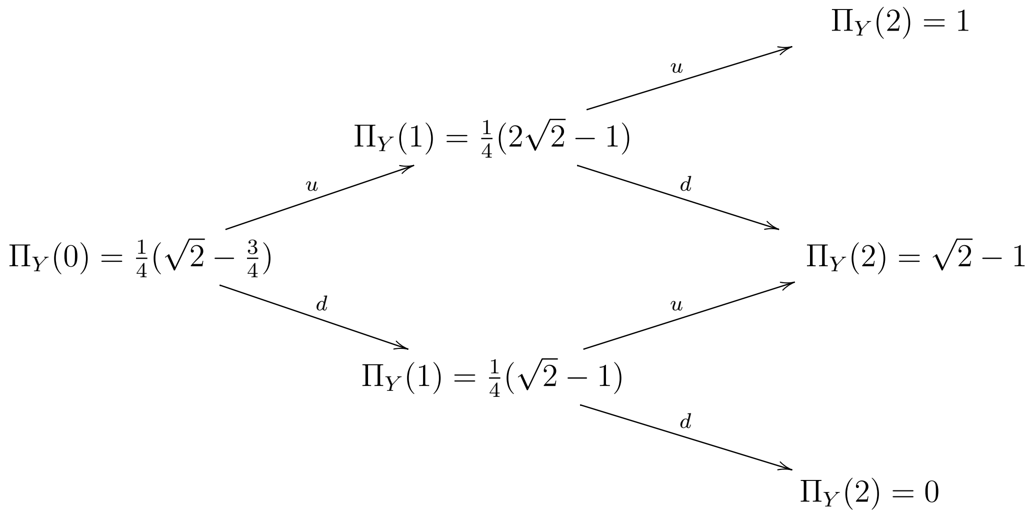
we have, at time $t = 1$,

$$S(1) = 2 \Rightarrow \Pi_Y(1) = \frac{3}{4}(\frac{1}{3} \cdot 1 + \frac{2}{3}(\sqrt{2} - 1)) = \frac{1}{4}(2\sqrt{2} - 1)$$
$$S(1) = 1 \Rightarrow \Pi_Y(1) = \frac{3}{4}(\frac{1}{3}(\sqrt{2} - 1) + \frac{2}{3} \cdot 0) = \frac{1}{4}(\sqrt{2} - 1)$$

while at time $t = 0$ we have

$$\Pi_Y(0) = \frac{3}{4}(\frac{1}{3} \cdot \frac{1}{4}(2\sqrt{2} - 1) + \frac{2}{3} \cdot \frac{1}{4}(\sqrt{2} - 1)) = \frac{1}{4}(\sqrt{2} - \frac{3}{4}).$$

Hence we have found the following diagram for the binomial price of the derivative




As to the probability that the derivative expires in the money, i.e., $\mathbb{P}(Y > 0)$, we see from the above diagram that this happens along the paths $(u, u), (u, d), (d, u)$, hence

$$\mathbb{P}(Y > 0) = \mathbb{P}(S^{(u,u)}) + \mathbb{P}(S^{(u,d)}) + \mathbb{P}(S^{(d,u)}) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16},$$

which corresponds to $43,75\%$.

**Remark 3.4.** Note carefully that the probability that the derivatives expires in the money is computed with the physical probability $p$ and *not* with the risk-neutral probability.

### 3.1.2 Example: A non-standard European derivative

Consider a 3-period binomial market with the following parameters:

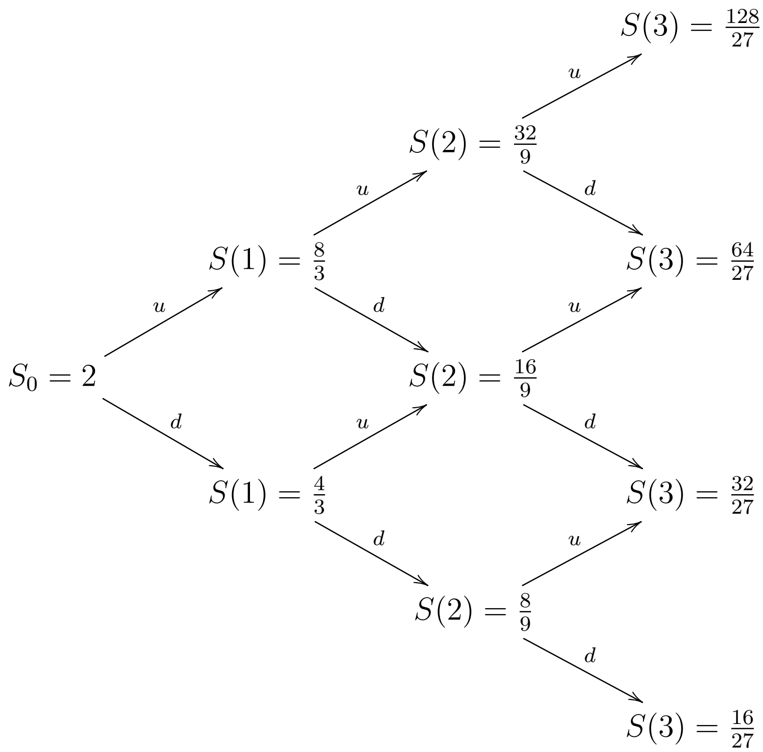
$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p = \frac{3}{4}.$$

Assuming $S_0 = 2$ and $r = 0$, compute the initial binomial price of the European derivative with pay-off

$$Y = \left(\frac{11}{9} - \min(S_0, S(1), S(2), S(3))\right)_+, \quad (z)_+ = \max(0, z),$$

and time of maturity $T = 3$. This is an example of **lookback option**. Compute the probability that the derivative expires in the money and the probability that the return of a constant portfolio with a long position on this derivative be positive.

**Solution:** We start by writing down the binomial tree of the stock price




To compute the initial binomial price of non-standard European derivatives it is convenient to use the formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x), \tag{3.9}$$

where $Y(x)$ denotes the pay-off as a function of the path of the stock price, $N_u(x)$ is the number of times that the stock price goes up in the path $x$ and $N_d(x) = N - N_u(x)$ is the number of times that it goes down. In this example we have $N = 3$, $r = 0$ and

$$q_u = q_d = \frac{1}{2}.$$

So, it remains to compute the pay-off for all possible paths of the binomial stock price, where

$$Y = \left( \frac{11}{9} - \min(S_0, S(1), S(2), S(3)) \right)_+, \quad (z)_+ = \max(0, z).$$

For instance

$$Y(u,u,u) = \left( \frac{11}{9} - \min(2, 8/3, 32/9, 128/27) \right)_+ = \left( \frac{11}{9} - 2 \right)_+ = \max(0, -\frac{7}{9}) = 0.$$

Similarly we find

$$
\begin{aligned}
Y(u,u,d) &= \left(\frac{11}{9} - \min(2, 8/3, 32/9, 64/27)\right)_+ = 0\\
Y(u,d,u) &= \left(\frac{11}{9} - \min(2, 8/3, 16/9, 64/27)\right)_+ = 0\\
Y(u,d,d) &= \left(\frac{11}{9} - \min(2, 8/3, 16/9, 32/27)\right)_+ = 1/27\\
Y(d,u,u) &= \left(\frac{11}{9} - \min(2, 4/3, 16/9, 64/27)\right)_+ = 0\\
Y(d,u,d) &= \left(\frac{11}{9} - \min(2, 4/3, 16/9, 32/27)\right)_+ = 1/27\\
Y(d,d,u) &= \left(\frac{11}{9} - \min(2, 4/3, 8/9, 32/27)\right)_+ = 1/3\\
Y(d,d,d) &= \left(\frac{11}{9} - \min(2, 4/3, 8/9, 16/27)\right)_+ = 17/27
\end{aligned}
$$

Replacing in (3.9) we obtain

$$\Pi_Y(0) = q_u(q_d)^2 Y(u,d,d) + (q_d)^2 q_u Y(d,u,d) + (q_d)^2 q_u Y(d,d,u) + (q_d)^3 Y(d,d,d),$$

the other terms being zero. Hence

$$\Pi_Y(0) = \frac{1}{8}\Big(\frac{1}{27} + \frac{1}{27} + \frac{1}{3} + \frac{17}{27}\Big) = \frac{7}{54}.$$

The probability that the derivative expires in the money is the probability that $Y > 0$. Hence we just sum the probabilities of the paths which lead to a positive pay-off:

$$
\begin{aligned}
\mathbb{P}(Y > 0) &= \mathbb{P}(S^{(u,d,d)}) + \mathbb{P}(S^{(d,u,d)}) + \mathbb{P}(S^{(d,d,u)}) + \mathbb{P}(S^{(d,d,d)})\\
&= p(1-p)^2 + (1-p)^2 p + (1-p)^2 p + (1-p)^3\\
&= 3(1-p)^2 p + (1-p)^3 = 3\left(\frac{1}{4}\right)^2 \frac{3}{4} + \left(\frac{1}{4}\right)^3 = \frac{5}{32} \approx 15,6\%
\end{aligned}
$$

Next consider a constant portfolio with a long position on the derivative. This means that we buy the derivative at time $t = 0$ and we wait (without changing the portfolio) until the expiration time $t = 3$. The return will be positive (i.e., the buyer makes a profit) if and only if $\Pi_Y(3) > \Pi_Y(0)$. But $\Pi_Y(3) = Y$, which, according to the computations above, is greater that $\Pi_Y(0) = 7/54$ only when the binomial stock price follows one of the paths $(d,d,u)$ or $(d,d,d)$. Hence

$$\mathbb{P}[R > 0] = \mathbb{P}(S^{(d,d,u)}) + \mathbb{P}(S^{(d,d,d)}) = (1-p)^2 p + (1-p)^3 = (1-p)^2 = \frac{1}{16} \approx 6,2\%$$

**Exercise 3.2** (•). *A* **compound option** *is an option whose underlying is another option. For instance, given $T_2 > T_1 > 0$ and $K_1, K_2 > 0$, a* **call on a put** *with maturity $T_1$ and strike $K_1$ is a contract that gives to its owner the right to buy at time $T_1$ for the price $K_1$ the put option on the stock with maturity $T_2$ and strike $K_2$. Let $S(t)$ be the price of the underlying stock of the put option. Assume that $S(t)$ follows a 2-period binomial model with parameters*

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}, \quad p = \frac{1}{4}, \quad S(0) = 16.$$

*Assume further that $T_2 = 2$, $T_1 = 1$, $K_1 = \frac{23}{9}$, $K_2 = 12$. Compute the initial price of the call on the put. Compute also the probability of positive return for the owner of the call on the put.*

**Exercise 3.3** (•). *Suppose $u > r > 0 \geq d$. A non-standard European derivative with maturity time $N$ has pay-off $Y = S(N)$ if $S(0) < S(1) < \cdots < S(N)$ and $Y = S(0)$ otherwise. Find $\Pi_Y(0)$.*

**Exercise 3.4** ($\star$). *A* **barrier option** *is an option that expires worthless as soon as the stock price exceeds (or fall below) a specified level (the barrier of the option[2]). For example, consider the binomial market in Section 3.1.2 and the barrier call option with strike $K = 2$ and barrier $B = 3$. This means that the derivative expires worthless if $S(3) \leq 2$ or if $S(t) > 3$ at some time $t \in \{1, 2, 3\}$. Compute the initial price $\Pi_Y(0)$ of this barrier option and the probability that it expires in the money.*

**Exercise 3.5** (•). *A European derivative with expiration $T = N$ pays the amount $Y = \log(S(T)/S(0))$. Find $\Pi_Y(0)$. Hint: Use the identity*

$$\binom{N}{k} k = N \binom{N-1}{k-1}.$$

*ANSWER: $\Pi_Y(0) = Te^{-rT}(q_u u + q_d d)$.*

**Exercise 3.6** (?). *Can you think of a reason to buy a call on the put, a barrier option, or a look-back option?*

## 3.2 Hedging portfolio

Next we treat the important problem of building a self-financing hedging portfolio for European derivatives.

**Theorem 3.3.** *Consider the European derivative with pay-off $Y$ at the time of maturity $T = N$. Then the portfolio process given by*

$$h_S(0) = h_S(1), \quad h_B(0) = h_B(1) \tag{3.10a}$$

[2] To be more precise, the definition just given refers to a knock-out barrier option with American barrier. There exists several variants (google it!)

*and, for $t \in \mathcal{I}$,*

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}, \tag{3.10b}$$

$$h_B(t) = \frac{e^{-r}}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d} \tag{3.10c}$$

*is a self-financing, predictable, hedging portfolio process.*

*Proof.* We first show that the given portfolio hedges the derivative. We have

$$V(t) = h_S(t)S(t) + h_B(t)B(t) = \frac{S(t)}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d} + \frac{e^{-r}B(t)}{B(t-1)} \frac{e^u \Pi_Y^d(t) - e^d \Pi_Y^u(t)}{e^u - e^d}.$$

Note that $e^{-r}B(t)/B(t-1) = 1$, while $S(t)/S(t-1)$ is either $e^u$ or $e^d$. By straightforward calculations we obtain $V^u(t) = \Pi_Y^u(t)$ and $V^d(t) = \Pi_Y^d(t)$, that is $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$. Hence the portfolio process is replicating, and therefore also hedging, the derivative. As to the self-financing property, we have

$$\begin{aligned} h_S(t)S(t-1) + h_B(t)B(t-1) &= \frac{\Pi_Y^u(t)(1 - e^{d-r}) + \Pi_Y^d(t)(e^{u-r} - 1)}{e^u - e^d} \\ &= e^{-r}(q_u \Pi_Y^u(t) + q_d \Pi_Y^d(t)) = \Pi_Y(t-1), \end{aligned}$$

where we used the definition of $q_u, q_d$, as well as the recurrence formula (3.8). By the already proven fact that $V(t) = \Pi_Y(t)$, for all $t \in \mathcal{I}$, we have

$$h_S(t)S(t-1) + h_B(t)B(t-1) = V(t-1),$$

which proves the self-financing property. Finally we show that the portfolio is predictable. Assume first that the European derivative is standard, i.e., $Y = g(S(N))$. Then $\Pi_Y(t) = f_g(t, S(t))$, see (3.5), and therefore

$$\Pi_Y^u(t) = f_g(t, S(t-1)e^u), \quad \Pi_Y^d(t) = f_g(t, S(t-1)e^d),$$

i.e., $\Pi_Y^u(t)$ and $\Pi_Y^d(t)$ are deterministic functions of $S(t-1)$. It follows that $h_S(t), h_B(t)$ given by (3.10b)-(3.10c) are also deterministic functions of $S(t-1)$, and so this portfolio process is predictable. In the case of non-standard derivatives we have similarly, by Exercise 3.1, that $\Pi_Y^u(t)$ and $\Pi_Y^d(t)$ are deterministic functions of $S(0), \dots, S(t-1)$, which again implies that the portfolio (3.10b)-(3.10c) is predictable. □

**Example.** Let us compute the hedging portfolio (3.10b)-(3.10c) for the standard derivative in Section 3.1.1. When $S(1) = 2$ we have $\Pi_Y^u(2) = 1$ and $\Pi_Y^d(2) = \sqrt{2} - 1$, hence (3.10b) gives

$$h_S(2) = \frac{1}{S(1)} \frac{1 - (\sqrt{2} - 1)}{2 - 1} = \frac{2 - \sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2} > 0 \quad \text{(long position)}.$$

When $S(1) = 1$ we have $\Pi_Y^u(2) = \sqrt{2} - 1$ and $\Pi_Y^d(2) = 0$, hence

$$h_S(2) = \frac{1}{S(1)} \frac{(\sqrt{2} - 1) - 0}{2 - 1} = \sqrt{2} - 1 > 0 \quad \text{(long position)}.$$

Recall that $h_S(2)$ is the position in the stock in the interval $(1, 2]$. In the interval $[0, 1]$ we have

$$h_S(0) = h_S(1) = \frac{1}{S(0)} \frac{\frac{1}{4}(2\sqrt{2} - 1) - \frac{1}{4}(\sqrt{2} - 1)}{2 - 1} = \frac{\sqrt{2}}{4} > 0 \quad \text{(long position)}.$$

Note that the seller, in this example, always keeps a long position on the stock in the hedging portfolio. The (short) position on the risk-free asset can be computed likewise using (3.10c).

**Exercise 3.7** (•). *Consider a 3-period binomial market with the following parameters:*

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1 \quad p = \frac{1}{2}.$$

*Assume $S_0 = \frac{64}{25}$. Consider the European derivative expiring at time $T = 3$ and with pay-off*

$$Y = S(3)H(S(3) - 1),$$

*where $H$ is the Heaviside function: $H(x) = 0$, if $x \geq 0$, $H(x) = 1$ if $x > 0$ (this is an example of* **cash-settled digital option***). Compute the possible paths of the derivative price and for each of them give the number of shares of the underlying asset in the hedging portfolio process. Compute the probability that the return of a constant portfolio with a short position in the derivative be positive.*

## 3.3 Computation of the binomial price of standard European derivatives with Matlab

As in Section 2.4, we consider a partition $0 = t_1 < t_2 < \cdots < t_{N+1} = T$ of the interval $[0, T]$ and the binomial stock price

$$S(i+1) = \begin{cases} S(i)e^u, & \text{with probability } p \\ S(i)e^d, & \text{with probability } 1 - p \end{cases}, \quad i \in \mathcal{I} = \{1, \dots, N\},$$

where $S(i) = S(t_i)$ and

$$u = \alpha h + \sigma \sqrt{\frac{1-p}{p}} \sqrt{h}, \quad d = \alpha h - \sigma \sqrt{\frac{p}{1-p}} \sqrt{h}, \quad h = \frac{T}{N}. \tag{3.11}$$

The value of the risk-free asset at time $t_i$ is given by $B(t_i) = B_0 e^{r t_i} = B_0 e^{(rh)i} := B(i)$. Hence the pair $(S(i), B(i))$ defines a 1+1 dimensional binomial market, in the sense of Section 2.2, with parameters $u, d$ given by (3.11) and interest rate $rh$. Note that

$$\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h} < rh < \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}$$

holds for $h$ small, hence the market is arbitrage free provided we take our time partition to be sufficiently fine. The recurrence formula (3.8) for the price of the European option with pay-off $Y$ at maturity $T$ becomes

$$\Pi_Y(N+1) = Y, \quad \text{and} \quad \Pi_Y(i) = e^{-rh}[q_u \Pi_Y^u(i+1) + q_d \Pi_Y^d(i+1)], \quad \text{for } t \in \mathcal{I},$$

where $\Pi_Y(i) = \Pi_Y(t_i)$ and

$$q_u = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}, \quad q_d = 1 - q_u.$$

The recurrence formula for standard European options is implemented by the Matlab function *BinomialEuropean* defined by the following code:

```
function P=BinomialEuropean(Q,S,r,g)
h=Q(2)-Q(1);
syms x;
f = sym(g);
N=length(Q)-1;
expu=S(1,2)/S(1,1);
expd=S(2,2)/S(1,1);
qu=(exp(r*h)-expd)/(expu-expd);
qd=(expu-exp(r*h))/(expu-expd);
if (qu<0 || qd<0)
display('Error:  the market is not arbitrage free.');
P=0;
return
end
P=zeros(N+1);
P(:,N+1)=subs(f,x,S(:,N+1));
for j=N:-1:1
for i=1:j
P(i,j)=exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1));
end
end
```

The arguments are the partition $Q = \{t_1, \dots, t_{N+1}\}$ and the binomial tree $S$ for the price of the underlying stock, computed with the function *BinomialStock* defined in Section 2.4, the interest rate $r$ of the money market and the pay-off function $g$ (e.g., `'max(x-10,0)'` for a call with strike $K = 10$). The function returns an upper-triangular $(N+1)\times(N+1)$ matrix which contains the binomial tree for the fair price of the derivative. The column $j$ contains the possible prices of the derivative at time $t_j$. A path of the derivative price is obtained by moving from each column to next one by either stays in the same row (which means that the price of the underlying stock went up at this step) or going down one row (which means that the price of the underlying went down at this step). Note that the Matlab function also checks that $(q_u, q_d)$ defines a probability, i.e., that the market is arbitrage free. If not, the function stops.

For example, let $S$ be the binomial tree (2.20) and run the command

```
P=BinomialEuropean(Q,S,0.01,'max(x-10,0)')
```

which computes the binomial price of a European call with strike $K = 10$. The result is

$$P = \begin{matrix} 0.2461 & 0.3817 & 0.5705 & 0.8141 & 1.0971 & 1.3875 \\ 0 & 0.1140 & 0.1978 & 0.3333 & 0.5387 & 0.8144 \\ 0 & 0 & 0.0325 & 0.0658 & 0.1333 & 0.2701 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad (3.12)$$

Observe that if the price of the stock goes down in the first three steps, then the price of the call becomes zero and remains zero for all subsequent times, regardless of the future path of the stock price. This happens because the binomial model predicts that the call has no chance to expire in the money when the stock price goes down in the first three steps and hence the call becomes worthless. This of course is in contradiction with reality, as the market price of calls is never zero prior to expire[3]. This paradox of the binomial model becomes less and less important the higher is the number of steps used for the computation. On the one hand if $N$ is too low the contribution of the paths where the call price is zero is significant for the computation of the fair price of the call and will determine an underestimation of this price. On the other hand as $N \to +\infty$ the binomial price of the call converges to the Black-Scholes price (see Chapter 6), and within the Black-Scholes theory the fair price of a call is always positive before maturity. Moreover it can be shown that the binomial algorithm to compute the fair price of European derivatives is stable in the following sense: the larger is the number of steps, the smaller is the numerical error due to truncations. In conclusion, *one can trust the results given by the binomial model only if $N$ is sufficiently large.* As a way of example, Figure 3.1 shows the initial binomial price of a call as a function of $N = 2, \dots, 100$. It is clear that for a small number of steps the result is unreliable, while for sufficiently many steps, say $N \gtrsim 20$, the result becomes stable.

Figures 3.2 and 3.3 show the initial price of the call as a function of the parameters $\alpha$ and $\sigma$. We see that (i) the price of the call is very weakly dependent on the mean of log-return $\alpha$

[3]In fact in real life we can never exclude with certainty that the call will expire in the money, irrespective of how deeply out of the money is the call today.

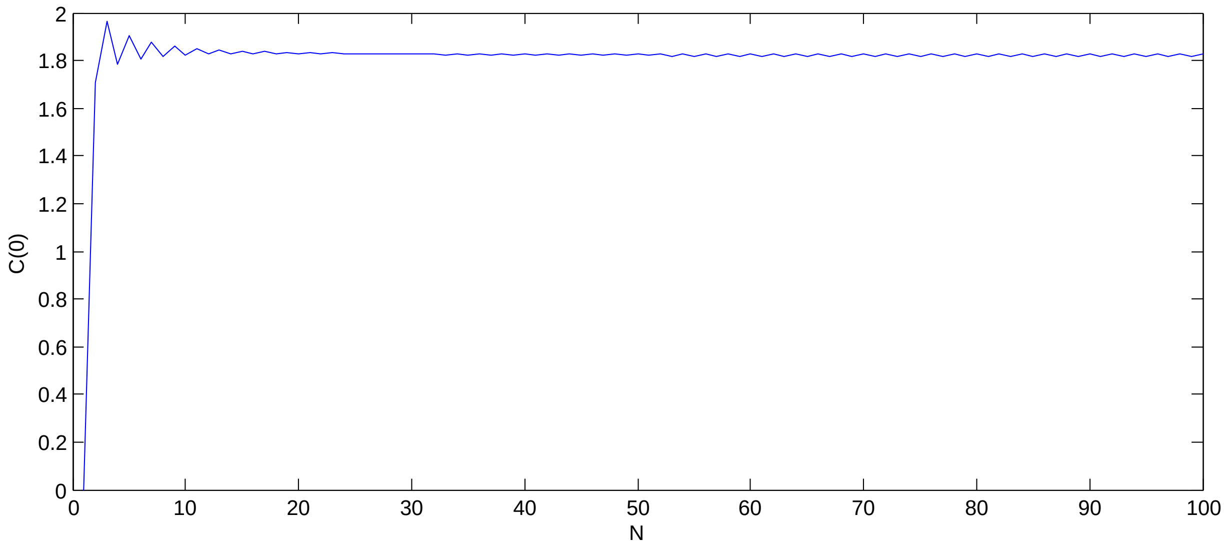



Figure 3.1: Initial price $C(0) = P(1,1)$ of the call computed for increasing values of $N$ ($S_0 = 10, K = 10.5, T = 1, \alpha = 0.1, \sigma = 0.5, p = 1/2$). The price stabilizes around 1.8 for $N \gtrsim 20$.

of the stock and (ii) the price of the call increases with the volatility of the stock. Hence the price of the call is not sensitive to the average movement of the stock price, but rather only to how uncertain is the stock price, measured by its volatility. In other words, according to the binomial model, the seller should not charge more for a call option if the stock price is expected to increase in the future, while rather the premium increases if the price it expected to be highly volatile. The analytical proof of these results is more easily carried out in the Black-Scholes model, which is the time-continuum limit of the binomial model (i.e., $N \to \infty, h \to 0$ such that $Nh = T$); see Chapter 6.

**Exercise 3.8** (?). *Can you give an intuitive explanation for the found dependence on the parameter $\alpha, \sigma$ of the call option binomial price?*

**Exercise 3.9.** *Let $N = 100$ and consider a European call.*

1. *Plot the initial binomial price of the call as a function of $S_0$*
2. *Plot the initial binomial price of the call as a function of the strike price*
3. *Plot the initial binomial price of the call as a function of the interest rate $r$*
4. *Verify the validity of the put-call parity*

**Exercise 3.10.** *Write a Matlab function which computes the hedging portfolio of standard European derivatives.*

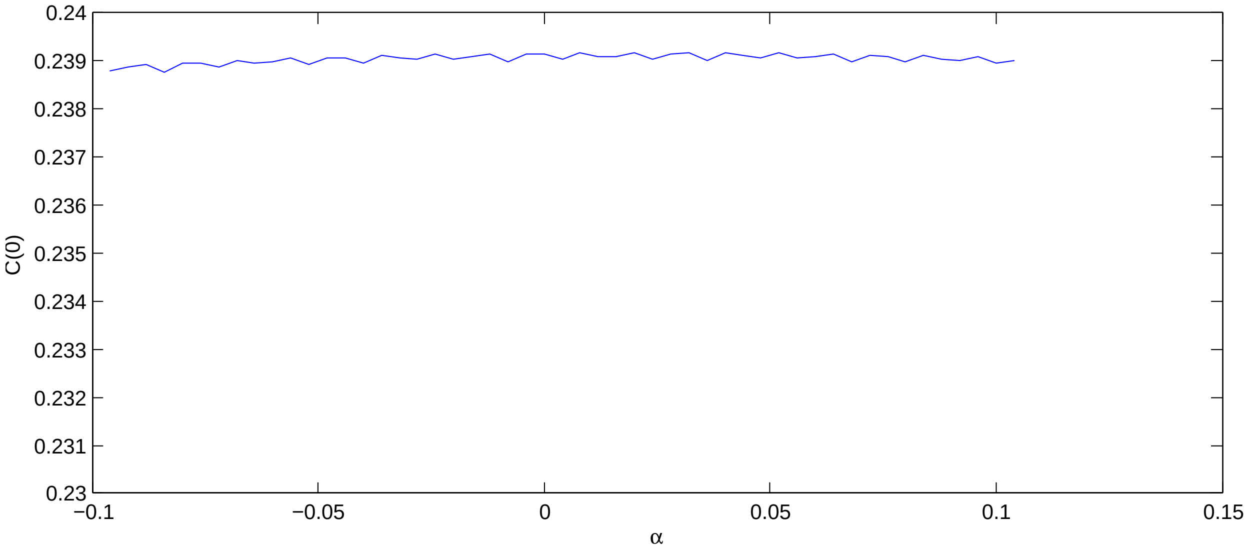



Figure 3.2: Initial price $C(0)$ of the call computed for increasing values of $\alpha \in [-0.1, 0.1]$ ($S_0 = 10, K = 10.5, T = 1, \sigma = 0.1, N = 1000, p = 1/2$). Clearly, the dependence on $\alpha$ of the call price is extremely weak.

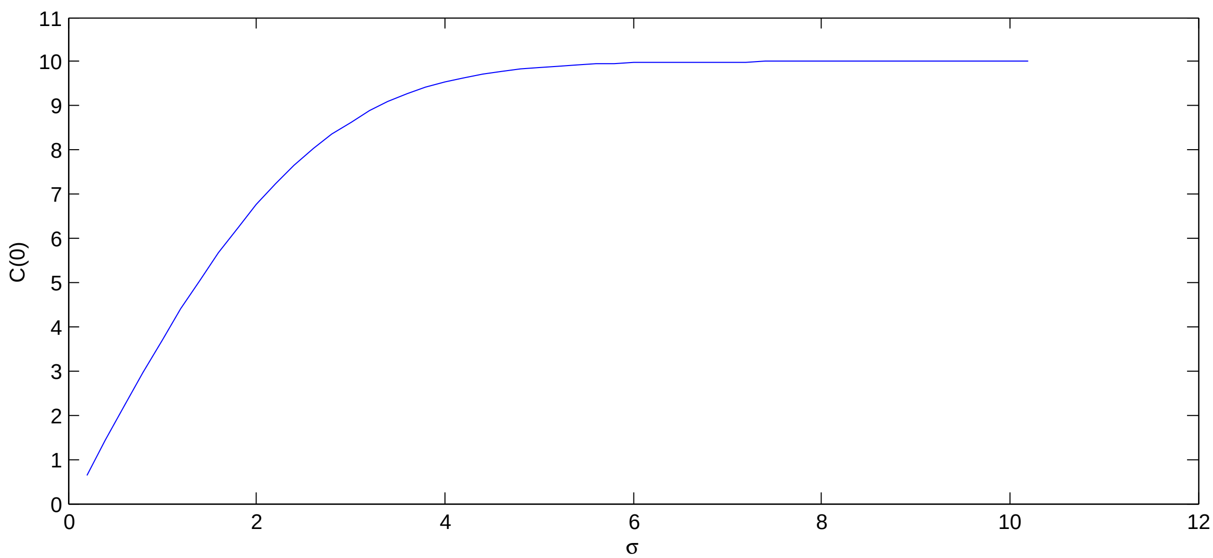



Figure 3.3: Initial price $C(0)$ of the call computed for increasing values of $\sigma > 0$ ($S_0 = 10, K = 10.5, T = 1, \alpha = 0, N = 1000, p = 1/2$). The call price increases with the volatility and approaches the stock price $S_0 = 10$ for $\sigma$ large (note however that only values $0 < \sigma \lesssim 2$ are realistic).

**Exercise 3.11.** *Look for the market price of call options on S&P500 which expire in the third Friday from now (the expiration date is always a Friday). Select 20 prices, 10 for the first options in the money and 10 for the first options out of the money. Let $S_0$ be the current value of S&P500 and $\sigma_{20}$ its current 20-days volatility. Compile a table with the following information: the first column contains the strike price $K$, the second column the market price, the third column the binomial price. Use $\sigma = \sigma_{20}$ and $r = 0$ to compute the binomial price, but check that the result does not change significantly for, say, $0 \leq r \leq 0.05$ (which is a quite large value for the interest rate). Explain why. Plot the difference between the market price and the theoretical price as a function of $K$.*

The difference between the theoretical fair price and the market price of a call option is commonly expressed in terms of the **implied volatility** $\sigma_{\mathrm{imp}}$ of the call, which is defined as the value of the parameter $\sigma$ to be used in the calculation of the binomial price in order for the latter to be equal to the market price[4].

**Exercise 3.12.** *Compute numerically the implied volatility of the options analysed in Exercise 3.11. Compare your value of $\sigma_{\mathrm{imp}}$ with the one quoted in the market. Plot the implied volatility as a function of the strike price and discuss your findings.*

We shall discuss again the important concept of implied volatility in Chapter 6.

[4]Actually, the computation of the implied volatility is typically done using the Black-Scholes model. However, as already mentioned, the binomial price and the Black-Scholes price are practically the same for $N$ sufficiently large.

# Chapter 4

# American derivatives

This chapter is concerned with American derivatives on a stock. We assume that the stock price follows the binomial model

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \quad t \in \mathcal{I} = \{1, \dots, N\},$$

where $0 < p < 1$, $u > d$. It is assumed that $S(0) = S_0$ is known. Moreover we assume that the interest rate $r$ of the money market is constant, so that the value of the risk-free asset at time $t$ is

$$B(t) = B_0 e^{rt}.$$

We impose $d < r < u$. In particular, the binomial market is arbitrage-free and the risk-neutral probability is well defined:

$$q_u = \frac{e^r - e^d}{e^u - e^d}, \quad q_d = \frac{e^u - e^r}{e^u - e^d}, \qquad q_u, q_d \in (0,1),\ q_u + q_d = 1.$$

We denote by $\{(h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ a portfolio process invested in the stock and the risk-free asset, where $(h_S(t), h_B(t))$ is the portfolio position in the interval $(t-1, t]$ and $h_S(0) = h_S(1)$, $h_B(0) = h_B(1)$. The value at time $t$ of the portfolio is $V(t) = h_S(t)S(t) + h_B(t)B(t)$. Recall that the portfolio process is said to be self-financing if

$$V(t-1) = h_S(t)S(t-1) + h_B(t)B(t-1), \quad t \in \mathcal{I},$$

which means that no cash is ever withdrawn or added to the portfolio. Recall also that the portfolio process is called predictable if there exist $N$ functions $H_1, \dots H_N$ such that $H_t : (0,\infty)^t \to \mathbb{R}^2$ and

$$(h_S(t), h_B(t)) = H_t(S(0), \dots, S(t-1)).$$

Another way to say this is that the portfolio position at time $t$ is a deterministic function of the stock price up to time $t-1$. In particular the portfolio position in the interval $(t-1, t]$ is determined by the information available up to and included the time $t-1$.

We shall need the proven recurrence formula for the binomial fair price $\Pi_Y(t)$ of European derivatives. Namely, denoting $\Pi_Y^u(t)$ the value of the European derivative at time $t$ assuming that the stock price goes up at time $t$ (i.e., $S(t) = S(t-1)e^u$, or equivalently, $x_t = u$), and similarly for $\Pi_Y^d(t)$, with "up" replaced by "down", we have seen in Theorem 3.2, that $\Pi_Y(t)$ satisfies

$$\Pi_Y(N) = Y, \quad \text{and} \quad \Pi_Y(t) = e^{-r}[q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)], \quad \text{for } t \in \{0, N-1\}. \tag{4.1}$$

## 4.1 The binomial price of American derivatives

In contrast to European derivatives, American derivatives can be exercised at any time prior or including the expiration date. Let $Y(t)$ be the pay-off of an American derivative exercised at time $t$. We assume that $t \in \mathcal{I} = \{1, \dots, N\}$, while $t = 0$ is the present time. Let $S(t)$ be the binomial stock price of the underlying stock at time $t$. We restrict ourselves to **standard American derivatives**, which means that

$$Y(t) = g(S(t)), \quad t \in \{0, 1, \dots, N\},$$

where $g : (0, \infty) \to [0, \infty)$ is the pay-off function of the derivative[1]. For example, $g(z) = (z - K)_+$ for American call options and $g(z) = (K - z)_+$ for American put options, where $(z)_+ = \max(0, z)$ and $K$ is the strike price of the option. $Y(t)$ is also called **intrinsic value** of the derivative. If $Y(t) = 0$ then the derivative is out of the money at time $t$. The initial pay-off $Y(0)$ is also denoted by $Y_0$; as it depends only on the initial price $S_0 = S(0)$ of the stock, the value of $Y_0$ is known. Moreover $Y(t)$ is clearly path-dependent, i.e., $Y(t) = Y(t, x)$. As usual, $Y^u(t)$ denotes the intrinsic value at time $t$ assuming that the stock price goes up at time $t$ and similarly for $Y^d(t)$, with "up" replaced by "down".

Next we want to introduce a reasonable definition for the binomial fair price of the American derivative with intrinsic value $Y(t)$ and maturity $T = N$. We denote this price by $\widehat{\Pi}_Y(t)$, while $\Pi_Y(t)$ denotes the binomial price of the corresponding European derivative with pay-off $Y(N) = g(S(N))$ at maturity $T = N$. As already discussed in Chapter 1, any meaningful definition of fair price for American derivatives must satisfy the following: (a) $\widehat{\Pi}_Y(N) = Y(N)$, (b) $\widehat{\Pi}_Y(t) \geq Y(t)$ and (c) $\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$. Property (a) fixes the price of the American derivative at time $N$. Due to (b) and (c), a reasonable definition for the fair price of the American derivative at time $t = N - 1$ is

$$\widehat{\Pi}_Y(N-1) = \max(Y(N-1), \Pi_Y(N-1)). \tag{4.2}$$

Using the recurrence formula (3.8), we have

$$\Pi_Y(N-1) = e^{-r}(q_u \Pi_Y^u(N) + q_d \Pi_Y^d(N)) = e^{-r}(q_u \widehat{\Pi}_Y^u(N) + q_d \widehat{\Pi}_Y^d(N)),$$

[1]For **non-standard American derivatives**, the pay-off has the form $Y(t) = g_t(S(1), \dots, S(t))$, where $g_t : (0, \infty)^t \to [0, \infty)$.

where for the second equality we used that $\widehat{\Pi}_Y(N) = \Pi_Y(N)$. Hence (4.2) becomes

$$\widehat{\Pi}_Y(N-1) = \max[Y(N-1), e^{-r}(q_u\widehat{\Pi}_Y^u(N) + q_d\widehat{\Pi}_Y^d(N))].$$

This suggests to introduce the following definition:

**Definition 4.1.** *The binomial (fair) price $\widehat{\Pi}_Y(t)$ of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ at time $t \in \{0, 1, \dots, N\}$ is defined by the recurrence formula*

$$\widehat{\Pi}_Y(N) = Y(N) \tag{4.3}$$

$$\widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u\widehat{\Pi}_Y^u(t+1) + q_d\widehat{\Pi}_Y^d(t+1))], \quad t \in \{0, \dots, N-1\}. \tag{4.4}$$

**Exercise 4.1** (?)**.** *Suppose an investor buys the American derivative at time $N-1$. Since the derivative can only be exercised at time $t = N$, why is the price of the American derivative not the same as the corresponding European derivative? Namely, why $\widehat{\Pi}_Y(N-1) = \max(Y(N-1), \Pi_Y(N-1))$ and not $\widehat{\Pi}_Y(N-1) = \Pi_Y(N-1)$?*

**Exercise 4.2** (?)**.** *Why did we not define the binomial fair price of the American derivative as $\widehat{\Pi}_Y(t) = \max(Y(t), \Pi_Y(t))$?*

In Section 4.4 we shall give further support to the statement that $\widehat{\Pi}_Y(t)$ does actually define a fair price for American derivatives. For the moment we limit ourselves to show that Definition 4.1 captures the properties of American derivatives discussed in Chapter 1.

**Theorem 4.1** ($*$)**.** *Let $\widehat{\Pi}_Y(t)$, $t \in \mathcal{I}$, denote the binomial price of the standard American derivative with intrinsic value $Y(t) = g(S(t))$ and maturity $T = N$. Let $\Pi_Y(t)$ denote the binomial price of the corresponding European derivative with pay-off $Y(N) = g(S(N))$ at maturity $N$. The following holds:*

*(a)* *$\widehat{\Pi}_Y(t)$ is a deterministic function of $S(t)$.*

*(b)* *$\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$, for all $t \in \mathcal{I}$.*

*(c)* *Assuming $r \geq 0$, the American call and the European call with the same strike and maturity have the same binomial price.*

*Proof.* (a) We have to show that $\widehat{\Pi}_Y(t) = H_t(S(t))$ for some functions $H_t : (0, \infty) \to \mathbb{R}$. We argue by induction. First this is true at time $N$, because $\widehat{\Pi}_Y(N) = Y(N) = g(S(N))$, hence $H_N = g$. Now assume that the claim is true at time $t+1$, i.e., there exists $H_{t+1} : (0, \infty) \to \mathbb{R}$ such that $\widehat{\Pi}_Y(t+1) = H_{t+1}(S(t+1))$. Hence

$$\widehat{\Pi}_Y^u(t+1) = H_{t+1}(S(t)e^u), \quad \widehat{\Pi}_Y^d(t+1) = H_{t+1}(S(t)e^d).$$

Thus, using $Y(t) = g(S(t))$, we have

$$\widehat{\Pi}_Y(t) = \max(g(S(t)), e^{-r}(q_u H_{t+1}(S(t)e^u) + q_d H_{t+1}(S(t)e^d)) = H_t(S(t)),$$

where $H_t(z) = \max(g(z), e^{-r}(q_u H_{t+1}(ze^u) + q_d H_{t+1}(ze^d))$.

(b) Also in this case we can argue by induction. The claim is true at maturity $N$, since at this time the value of both derivatives equals the pay-off by definition. Now assume the claim is true at time $t+1$, i.e.,

$$\widehat{\Pi}_Y(t+1) \geq \Pi_Y(t+1).$$

Hence

$$e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1)) \geq e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)),$$

which implies

$$\begin{aligned}\widehat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1)) \geq e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1)) \\ &\geq e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)) = \Pi_Y(t),\end{aligned}$$

where we used that the price of European calls satisfies the recurrence relation (4.1)

(c) To prove that American and European calls with equal parameters have the same value, we first observe that it suffices to prove that $\widehat{\Pi}_{call}(t) \leq \Pi_{call}(t)$, since we already proved in (b) that $\widehat{\Pi}_{call}(t) \geq \Pi_{call}(t)$ holds (we prove this for all standard American derivatives!). We argue again by induction. At maturity $\widehat{\Pi}_{call}(N) = \Pi_{call}(N)$. Assume that $\widehat{\Pi}_{call}(t+1) \leq \Pi_{call}(t+1)$. Then

$$e^{-r}(q_u \widehat{\Pi}_{call}^u(t+1) + q_d \widehat{\Pi}_{call}^d(t+1)) \leq e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)). \tag{4.5}$$

Now let $\Pi_{put}(t)$ be the price of the European put with the same parameters as the European call. We have shown in Theorem 3.1 that the put-call parity holds:

$$\Pi_{call}(t) = S(t) - Ke^{-r(T-t)} + \Pi_{put}(t).$$

As $\Pi_{put}(t) \geq 0$ and $r \geq 0$ we have $\Pi_{call}(t) \geq S(t) - K$. Since $\Pi_{call}(t) \geq 0$, we have

$$\Pi_{call}(t) \geq \max(S(t) - K, 0) = (S(t) - K)_+ = Y(t),$$

where $Y(t)$ is the intrinsic value of the American call. As

$$\Pi_{call}(t) = e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)),$$

we have

$$Y(t) \leq e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)).$$

Hence, using that $\max(a,b) \leq \max(c,b)$ holds when $a \leq c$,

$$\begin{aligned}\widehat{\Pi}_{call}(t) &= \max(Y(t), e^{-r}(q_u \widehat{\Pi}_{call}^u(t+1) + q_d \widehat{\Pi}_{call}^d(t+1)) \\ &\leq \max(e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)), e^{-r}(q_u \widehat{\Pi}_{call}^u(t+1) + q_d \widehat{\Pi}_{call}^d(t+1))).\end{aligned}$$

By (B.5),

$$\begin{aligned}&\max(e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)), e^{-r}(q_u \widehat{\Pi}_{call}^u(t+1) + q_d \widehat{\Pi}_{call}^d(t+1))) \\ &\quad = e^{-r}(q_u \Pi_{call}^u(t+1) + q_d \Pi_{call}^d(t+1)) = \Pi_{call}(t).\end{aligned}$$

Hence $\widehat{\Pi}_{call}(t) \leq \Pi_{call}(t)$, and the proof is complete. □

## 4.2 Optimal exercise time of American put options

The last claim in Theorem 4.1 is consistent with what we have seen in Chapter 1, namely that it is never optimal to exercise an American call option prior to expiration when the underlying stock pays no dividends. The example treated in Section 4.3 below shows that this is not true for put options. To this regard we remark that, when the American put is priced using the binomial model, an optimal exercise time in the sense of Definition 1.1 is a time $t \in \mathcal{I}$ such that

$$(K - S(t))_+ = \max[(K - S(t))_+, e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1))],$$

that is to say, a time $t \in \mathcal{I}$ at which

$$(K - S(t))_+ \geq e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1)). \tag{4.6}$$

It can be shown that the inequality (4.6) holds at time $t \in \{1, \dots, N-1\}$ if and only if $S(t) \leq S_*(t)$, where $S_*(t) < K$ depends only on the market parameters. The exact expression for $S_*(t)$ is unknown, but it can be easily determined numerically, see Exercise 4.9. This fact is very important in the applications, for it tells us that the American put should be exercised as soon as the price of the stock falls below the value $S_*(t)$, irrespective of whether the market price of the put option equals its intrinsic value (which is extremely unlike to happen). In the next theorem the exact value of $S_*(t)$ is derived in the 2-period model.

**Theorem 4.2** (∗)**.** *In a 2-period binomial model with parameters $u > 0$, $d < 0$, $0 < r < u$, the earlier exercise at time $t = 1$ of the American put with strike $K$ and maturity $T = 2$ is optimal if and only if $S(1) \leq S_*(1)$, where*

$$S_*(1) = K \frac{1 - e^{-r} q_d}{1 - e^{-r} q_d e^d}. \tag{4.7}$$

*Proof.* We need to study for which values of $S(1)$ is (4.6) satisfied at time $t = 1$. Since $\widehat{\Pi}_Y^u(2) = (K - S(1)e^u)_+$ and $\widehat{\Pi}_Y^d(2) = (K - S(1)e^d)_+$, (4.6) becomes

$$(K - S(1))_+ \geq e^{-r}(q_u(K - S(1)e^u)_+ + q_d(K - S(1)e^d)_+). \tag{4.8}$$

We study the inequality (4.8) for separate values of $S(1)$ in the intervals

$$\begin{aligned} &S(1) \in [0, Ke^{-u}] := I_1, \ S(1) \in [Ke^{-u}, K] := I_2, \\ &S(1) \in [K, Ke^{-d}] := I_3, \ S(1) \in [Ke^{-d}, +\infty) := I_4. \end{aligned}$$

For $S(1) \in I_1$ we have,

$$e^{-r}(q_u(K - S(1)e^u)_+ + q_d(K - S(1)e^d)_+) = e^{-r}(q_u e^u(Ke^{-u} - S(1)) + q_d e^d(Ke^{-d} - S(1))).$$

Using $q_u + q_d = 1$ and $q_u e^u + q_d e^d = e^r$ we obtain

$$e^{-r}(q_u(K - S(1)e^u)_+ + q_d(K - S(1)e^d)_+) = K - S(1) = (K - S(1))_+, \quad \text{for } S(1) \in I_1.$$

Similarly, for $S(1) \in I_2$ we have

$$e^{-r}(q_u(K - S(1)e^u)_+ + q_d(K - S(1)e^d)_+) = \begin{cases} (K - S(1))_+ & \text{for } S(1) \leq S_*(1) \\ e^{-r}q_d e^d(Ke^{-d} - S(1)) & \text{for } S(1) > S_*(1) \end{cases}$$

where $S_*(1)$ is given by (4.7). Treating similarly the cases $S(1) \in I_3$ and $S(1) \in I_4$ we find that the price of the American put at time $t = 1$ is given by

$$\widehat{\Pi}(1) = \begin{cases} (K - S(1))_+ & \text{for } 0 < S(1) \leq S_*(1) \\ e^{-r}q_d e^d(Ke^{-d} - S(1)) & \text{for } S_* < S(1) \leq Ke^{-d} \\ 0 & \text{for } S > Ke^{-d} \end{cases}$$

We conclude that it is optimal to exercise the American put at time $t = 1$ if and only if $S(1) \leq S_*(1)$. □

## 4.3 Example of American put option

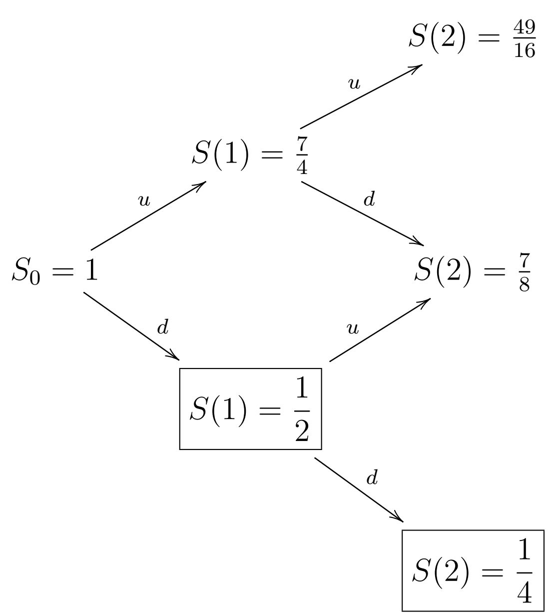
In this section we consider an example of American put option. We let the strike price $K = 3/4$, and so

$$Y(t) = \left(\frac{3}{4} - S(t)\right)_+ .$$

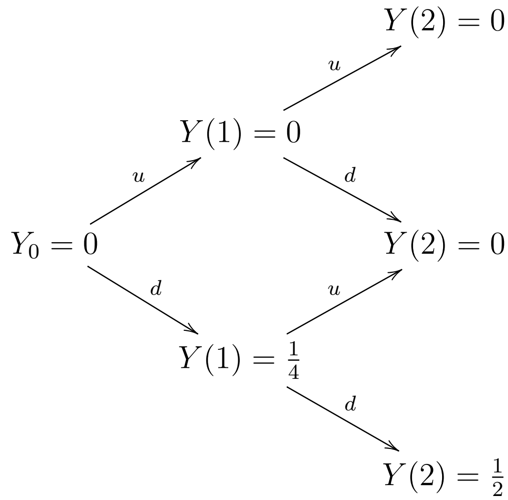
We present an example in $N = 2$ periods. Consider a market with the following parameters:

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8},$$

so that $q_u = q_d = 1/2$. Assuming $S_0 = 1$, the binomial tree for the stock price is



When the price of the stock in the paths above is within a box, the put option is in the money. In fact, the binomial tree for the intrinsic value $Y(t)$ of the American put is




Let us first compute the price of the corresponding European put with pay off

$$\left(\frac{3}{4} - S(2)\right)_+ = Y(2) = \Pi_Y(2)$$

at time of maturity $T = 2$. Using the recurrence formula (4.1) we have

$$\Pi_Y(1) = \frac{4}{9}[\Pi_Y^u(2) + \Pi_Y^d(2)] = \frac{4}{9}\left[\left(\frac{3}{4} - \frac{7}{4}S(1)\right)_+ + \left(\frac{3}{4} - \frac{1}{2}S(1)\right)_+\right].$$

Hence

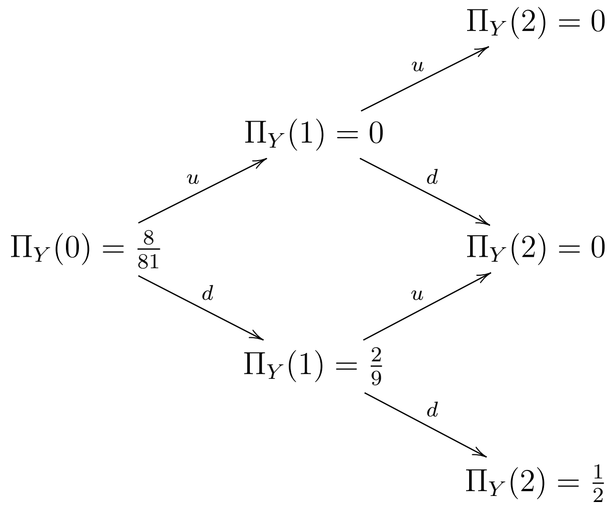
$$\Pi_Y^u(1) = \frac{4}{9}\left[\left(\frac{3}{4} - \frac{49}{16}\right)_+ + \left(\frac{3}{4} - \frac{7}{8}\right)_+\right] = 0,$$

$$\Pi_Y^d(1) = \frac{4}{9}\left[\left(\frac{3}{4} - \frac{7}{8}\right)_+ + \left(\frac{3}{4} - \frac{1}{4}\right)_+\right] = \frac{2}{9}.$$

Therefore, again by (4.1), we have

$$\Pi_Y(0) = \frac{4}{9}[\Pi_Y^u(1) + \Pi_Y^d(1)] = \frac{8}{81}.$$

In conclusion, we have obtained the following paths for the binomial price of the European derivative




**Exercise 4.3** (?)**.** *Let* $\{(h_S(t), h_B(t)), t = 0, 1, 2\}$ *be a hedging, self-financing portfolio process for this European put. Can you guess whether* $h_S(0)$ *will be positive or negative without computing the portfolio?*

Now we compute the prices of the American put option. Of course, at time of maturity it is the same as its European counterpart (by definition of hedging portfolio). At time $t = 1$ we have, by (4.4),

$$\begin{aligned}\widehat{\Pi}_Y(1) &= \max\left[Y(1), \frac{4}{9}(\widehat{\Pi}_Y^u(2) + \widehat{\Pi}_Y^d(2))\right] \\ &= \max\left[Y(1), \frac{4}{9}\left(\left(\frac{3}{4} - \frac{7}{4}S(1)\right)_+ + \left(\frac{3}{4} - \frac{1}{2}S(1)\right)_+\right)\right].\end{aligned}$$

Since

$$Y^u(1) = \left(\frac{3}{4} - \frac{7}{4}\right)_+ = 0, \quad Y^d(1) = \left(\frac{3}{4} - \frac{1}{2}\right)_+ = \frac{1}{4},$$

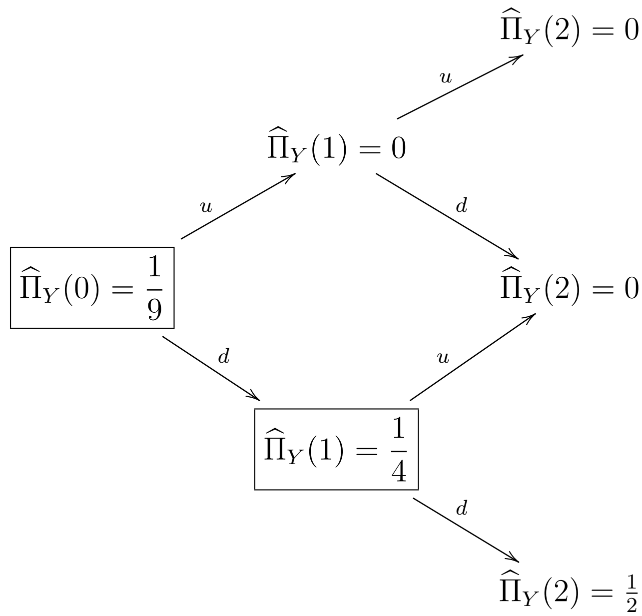
we find

$$\widehat{\Pi}_Y^u(1) = \max[0, 0] = 0, \quad \widehat{\Pi}_Y^d(1) = \max\left[\frac{1}{4}, \frac{2}{9}\right] = \frac{1}{4}$$

and so

$$\widehat{\Pi}_Y(0) = \max\left[Y(0), \frac{4}{9}(\widehat{\Pi}_Y^u(1) + \widehat{\Pi}_Y^d(1))\right] = \frac{1}{9}.$$

Hence the binomial price of the American put corresponding to the different paths of the stock price is as follows:




Note that the binomial price of the American put and of the European put are different in two instances, which are indicated in the paths above by putting the price of the American put within a box. In particular, their initial price is different. When the prices are different, the American put is more expensive than the European put. Moreover, if (and only if) the stock price goes down at time $t = 1$, the binomial price of the American put equals its intrinsic value prior to maturity, hence in this case (and only in this case) the earlier exercise at time $t = 1$ is optimal. This result agrees with Theorem 4.2, since the value (4.7) for this option is given by

$$S_*(1) = \frac{3}{4}\frac{1 - \frac{8}{9}\frac{1}{2}}{1 - \frac{8}{9}\frac{1}{2}\frac{1}{2}} = \frac{15}{28} \in \left(\frac{1}{2}, \frac{7}{4}\right).$$

## 4.4 Hedging portfolio processes of American derivatives

In this section we describe how to obtain hedging portfolio processes for American derivatives. Recall that for the European derivative with pay-off $Y$ at the expiration date $T = N$, hedging portfolio processes have been defined by imposing that their value $V(t)$ satisfies $V(N) = Y$, see Definition 3.2. Because of the earlier exercise option of American derivatives, this definition cannot be adopted in the American case. In fact, since the goal of an hedging portfolio is to secure the writer position (i.e., the short position) and the buyer of the American derivative has the right to exercise the derivative at any time $t \in \mathcal{I}$, then we need to ensure that the value of the writer portfolio is at *any* time—and not only at maturity—sufficient to pay-off the buyer, i.e., $V(t) \geq Y(t)$. This leads us to define hedging portfolio processes for American derivatives as follows.

**Definition 4.2.** *A portfolio process* $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ *is said to be hedging the American derivative with intrinsic value* $Y(t)$ *and maturity* $T = N$ *if*

$$V(N) = Y(N), \quad V(t) \geq Y(t),\ t = 0, \dots, N-1,$$

*where* $V(t) = h_S(t)S(t) + h_B(t)B(t)$ *is the value of the portfolio process at time* $t$*.*

Now, the most important hedging portfolios are those which **replicate** the American derivative, i.e., the portfolio processes whose value $V(t)$ satisfies $V(t) = \widehat{\Pi}_Y(t)$, for all $t \in [0, T]$. Note that replicating portfolio processes are hedging portfolios, because $\widehat{\Pi}_Y(t) \geq Y(t)$. In the European case any self-financing hedging portfolio process is (trivially) replicating, because $\Pi_Y(t)$ has been defined as the common value of any such portfolio. However, in the American case, the value at time $t$ of an hedging portfolio process could be strictly greater than the binomial price $\widehat{\Pi}_Y(t)$ of the American derivative given in Definition 4.1. If this happens, then the writer should withdraw cash from the portfolio in order to replicate the value of the American derivative. This leads us to introduce the important concept of portfolio processes generating a cash flow. We argue as we did for self-financing portfolios. Recall that $(h_S(t), h_B(t))$ is the investor position on the stock and the risk-free asset during the time interval $(t-1, t]$. Let $V(t) = h_S(t)S(t) + h_B(t)B(t)$ be the value of this portfolio. At the time $t$, the investor sells/buys shares of the two assets. Let $(h_S(t+1), h_B(t+1))$ be the new position on the stock and the risk-free asset in the interval $(t, t+1]$. Then the value of the portfolio process immediately after changing the position at time $t$ is given by $V'(t) = h_S(t+1)S(t) + h_B(t+1)B(t)$. The cash flow $C(t)$ is defined as $V'(t) - V(t) = -C(t)$ and corresponds to cash withdrawn (if $C(t) > 0$) or added (if $C(t) < 0$) to the portfolio as a result of the change in the position on the assets (for a self-financing portfolio we have of course $C(t) = 0$, for all $t \in \{0, \dots, N\}$). This leads to the following definition.

**Definition 4.3.** *A portfolio process* $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ *is said to generate the* **cash flow** $C(t-1)$*,* $t \in \mathcal{I}$*, if*

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1) - C(t-1), \quad t \in \mathcal{I},$$

*or, equivalently,*

$$V(t) - V(t-1) = h_S(t)(S(t) - S(t-1)) + h_B(t)(B(t) - B(t-1)) - C(t-1).$$

*In particular, if* $C(t-1) > 0$*, then the cash is withdrawn from the portfolio, causing a decrease of its value, while if* $C(t-1) < 0$*, then the cash is added to the portfolio, causing an increasing of its value.*

**Remark 4.1.** As we assume $h_S(0) = h_S(1)$ and $h_B(0) = h_B(1)$, then $C(0) = 0$. Therefore the first time at which the investor can add/remove cash from the portfolio is after changing the position (instantaneously) at time $t = 1$, i.e., when passing from $(h_S(1), h_B(1))$ to $(h_S(2), h_B(2))$, generating the cash flow $C(1)$.

**Example:** Consider a constant portfolio process that consists of only one share of the stock, that is $h_S(t) \equiv 1$ and $h_B(t) \equiv 0$. The value at time $t$ of this portfolio is $V(t) = h_S(t)S(t) = S(t)$. Suppose that in the interval of time $(t-1, t)$ the stock pays a dividend of 1%. This means that the fraction $S(t-1)/100$ is deposited into the account of the investor, while the price of the stock decreases of the same amount. Hence the value of the portfolio at time $t$ is

$$V(t) = S(t) - S(t-1)/100.$$

Therefore

$$V(t) - V(t-1) = S(t) - S(t-1)/100 - S(t-1) = h_S(t)(S(t) - S(t-1)) - C(t-1),$$

where $h_S(t) = 1$ and $C(t) = S(t-1)/100$.

We now show how to build a predictable replicating portfolio for American derivatives.

**Theorem 4.3.** *Consider the standard American derivative with intrinsic value $Y(t)$ and maturity $T = N$. Let $\widehat{\Pi}_Y(t)$ be its binomial price as given by (4.3)-(4.4). Define the portfolio process $\{(\widehat{h}_S(t), \widehat{h}_B(t))\}_{t\in\mathcal{I}}$ and the cash flow process $C(t)$ recursively as follows:*

$$C(0) = 0, \quad C(t-1) = \widehat{\Pi}_Y(t-1) - e^{-r}[q_u\widehat{\Pi}^u_Y(t) + q_d\widehat{\Pi}^d_Y(t)], \quad t \in \{2, \dots, N\}, \tag{4.9}$$

$$\widehat{h}_S(1) = \widehat{h}_S(0), \quad \widehat{h}_B(0) = \widehat{h}_B(1) \tag{4.10a}$$

*and, for $t = 1, \dots, N$,*

$$\widehat{h}_S(t) = \frac{1}{S(t-1)}\frac{\widehat{\Pi}^u_Y(t) - \widehat{\Pi}^d_Y(t)}{e^u - e^d}, \tag{4.10b}$$

$$\widehat{h}_B(t) = \frac{e^{-r}}{B(t-1)}\frac{e^u\widehat{\Pi}^d_Y(t) - e^d\widehat{\Pi}^u_Y(t)}{e^u - e^d}. \tag{4.10c}$$

*The portfolio process* (4.10) *is predictable, replicates the American derivative and generates the cash-flow* (4.9)*.*

*Proof.* We have shown in Theorem 4.1 that $\widehat{\Pi}_Y(t)$ is a deterministic function of $S(t)$, i.e., $\widehat{\Pi}_Y(t) = f_t(S(t))$, for some function $f_t : (0, \infty) \to (0, \infty)$. Hence

$$\widehat{\Pi}^u_Y(t) = f_t(S(t-1)e^u), \quad \widehat{\Pi}^d_Y(t) = f_t(S(t-1)e^d)$$

are deterministic functions of $S(t-1)$, by which it follows immediately that the portfolio process (4.10) is predictable. To show that the portfolio replicates the derivative and generates the cash flow (4.9) we have to show that

$$V(t) = \widehat{\Pi}_Y(t), \quad \text{for all } t \in \{0, \dots N\} \tag{4.11}$$

and

$$V(t-1) = \widehat{h}_S(t)S(t-1) + \widehat{h}_B(t)B(t-1) + C(t-1), \quad \text{for all } t \in \mathcal{I}. \tag{4.12}$$

The proof is straightforward: just replace (4.9) and (4.10) into (4.11)-(4.12). For instance, assuming that the price goes up at time $t$, we compute

$$\begin{aligned}
V^u(t) &= \widehat{h}_S(t)S(t-1)e^u + \widehat{h}_B(t)B(t-1)e^r \\
&= \left(\frac{\widehat{\Pi}_Y^u(t) - \widehat{\Pi}_Y^d(t)}{e^u - e^d}\right) e^u + \left(\frac{e^u\widehat{\Pi}_Y^d(t) - e^d\widehat{\Pi}_Y^u(t)}{e^u - e^d}\right) \\
&= \widehat{\Pi}_Y^u(t),
\end{aligned}$$

and at the same fashion one proves that $V^d(t) = \widehat{\Pi}_Y^d(t)$. Hence (4.11) holds. In a similar fashion, replacing (4.9) and (4.10) into the right hand side of (4.12) we find that the latter is equal to $\widehat{\Pi}_Y(t-1)$, which we already proved to be equal to $V(t-1)$. Hence (4.12) holds as well. □

The previous theorem is telling us that the writer can hedge the derivative and still be able to withdraw cash from the portfolio. While this might seem unfair from the buyer point of view, we remark that whether the writer is allowed or not to withdraw a positive amount of cash from the portfolio (i.e., $C(t) > 0$) depends on the "smartness" of the buyer. In fact, using (4.4) in (4.9), we have, for $t \in \{2, \dots, N\}$,

$$C(t-1) = \max(Y(t-1), e^{-r}(q_u\widehat{\Pi}_Y^u(t)) + q_d\widehat{\Pi}_Y^d(t)) - e^{-r}(q_u\widehat{\Pi}_Y^u(t)) + q_d\widehat{\Pi}_Y^d(t)).$$

This quantity is positive at time $t-1$ if and only if only $Y(t-1) > e^{-r}(q_u\widehat{\Pi}_Y^u(t)) + q_d\widehat{\Pi}_Y^d(t))$, which implies that $t-1$ is an optimal exercise time. Hence the writer of the American put can withdraw cash from the portfolio only if the buyer fails to exercise the derivative optimally. If however the buyer exercises the derivative optimally, then the seller needs the full value of the portfolio to pay-off the buyer and thus no cash can be withdrawn.

**Example.** Computing the cash flow at $t = 1$ for the American put considered in Section 4.3, we find

$$C(1) = \widehat{\Pi}_Y(1) - \frac{4}{9}\left(\left(\frac{3}{4} - \frac{7}{4}S(1)\right)_+ + \left(\frac{3}{4} - \frac{1}{2}S(1)\right)_+\right).$$

If the stock price goes up at time $t = 1$ we obtain $C^u(1) = 0$; if the stock price goes down at time $t = 1$ we obtain

$$C^d(1) = \frac{1}{4} - \frac{4}{9}\left(\left(\frac{3}{4} - \frac{7}{8}\right)_+ + \left(\frac{3}{4} - \frac{1}{4}\right)_+\right) = \frac{1}{36}.$$

Hence if at time $t = 1$ the price of the stock goes down and the buyer does not exercise the American put, then the writer can withdraw the cash $\frac{1}{36}$ from the portfolio. The value remaining in the portfolio is $\frac{1}{4} - \frac{1}{36} = \frac{2}{9}$. The American put can be hedged with this value by short selling $\frac{4}{5}$ shares of the stock and lending $(\frac{2}{9} + \frac{4}{5}\cdot\frac{1}{2}) = \frac{28}{45}$ in the money market. So doing the writer will receive the amount $\frac{28}{45}\cdot\frac{9}{8} = \frac{7}{10}$ at time $t = 2$; if the stock price goes up at time $t = 2$ the American put expires out of the money and the writer will use the amount $\frac{7}{10}$ to buy $\frac{4}{5}$ shares of the stock at the price $S(2) = \frac{7}{8}$ and close the short position on the stock without losses. If the price of the stock goes down at time $t = 2$, the writer will give the pay-off $\frac{1}{2}$ to the buyer of the American put and use the remaining value $\frac{7}{10} - \frac{1}{2} = \frac{1}{5}$ in the portfolio to buy $\frac{4}{5}$ shares of the stock at the price $S(2) = \frac{1}{4}$ and again close the short position on the stock without losses.

**Exercise 4.4** (•). *Consider the American derivative with intrinsic value*

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

*and expiring at time $T = 3$. The initial price of the underlying stock is $S(0) = 27$, while at future times it follows the binomial model*

$$S(t+1) = \begin{cases} 4S(t)/3 & \text{with probability } 1/2 \\ 2S(t)/3 & \text{with probability } 1/2 \end{cases}$$

*for $t = 0, 1, 2$. Assume also that the interest rate of the money market is zero. Compute the possible paths of the binomial price of the derivative. In which case it is optimal for the buyer to exercise the derivative prior to expiration? What is the amount of cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative optimally?*

**Exercise 4.5** (•). *Consider a 3-period binomial model with the following parameters:*

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad p \in (0, 1).$$

*Let $S(0) = \frac{64}{25}$ be the initial price of the stock. Consider an American style derivative on the stock with maturity $T = 3$ and intrinsic value*

$$Y(t) = \big|3 - S(t)\big|\, H\big(S(t) - 7/5\big),$$

*where $H(x)$ is the Heaviside function and $|x|$ is the absolute value of $x$ (recall that $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x \leq 0$). Compute the binomial price of the derivative at each time $t \in \{0, 1, 2, 3\}$ (max. 1 point) and the initial position on the stock in the hedging portfolio. Compute the cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative at optimal times. Compute the probability that the derivative is in the money at time $t$ and the probability that the return for the buyer is positive at time $t$, where $t \in \{0, 1, 2, 3\}$.*

**Exercise 4.6** (•). *Consider a 2-period binomial model with the same parameters as in Exercise 4.5. Compute the price at time $t \in \{0, 1, 2\}$ of the American put on the stock with maturity $T = 2$ and strike price $K_2 = \frac{11}{5}$ and identify the possible optimal exercise times prior to maturity. Next consider the compound option which gives to its owner the right to buy the American put at time $t = 1$ for the price $K_1 = \frac{8}{25}$. Compute the price of the compound option at time $t = 0$ and the hedging portfolio for the compound option (assume $B(0) = 1$). Compute the maximum expected return in the interval $t \in [0, 2]$ for the owner of the compound option as a function of $p \in (0, 1)$.*

**Exercise 4.7** ($\star$). *Compute the replicating portfolio for the American put given in Section 4.3 (assume $B_0 = 1$).*

## 4.5 Computation of the fair price of American derivatives with Matlab

We work under the same set-up as in Section 3.3. Namely we consider a partition $0 = t_1 < t_2 < \cdots < t_{N+1} = T$ of the interval $[0,T]$ and the binomial stock price

$$S(i+1) = \begin{cases} S(i)e^u, & \text{with probability } p \\ S(i)e^d, & \text{with probability } 1-p \end{cases}, \quad i \in \mathcal{I} = \{1,\dots,N\},$$

where $S(i) = S(t_i)$ and

$$u = \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}. \tag{4.13}$$

The value of the risk-free asset at time $t_i$ is given by $B(t_i) = B_0 e^{rt_i} = B_0 e^{(rh)i} := B(i)$. Hence the pair $(S(i), B(i))$ defines a 1+1 dimensional binomial market with parameters $u, d$ given by (4.13) and interest rate $rh$. The definition (4.1) of binomial price of American derivative becomes

$$\widehat{\Pi}_Y(N+1) = Y(N+1) \quad \widehat{\Pi}_Y(i) = \max(Y(i), e^{-rh}(q_u\widehat{\Pi}^u_Y(i+1)) + q_d\widehat{\Pi}^d_Y(i+1)), \quad t \in \mathcal{I},$$

where $\widehat{\Pi}_Y(i) = \widehat{\Pi}_Y(t_i)$ and

$$q_u = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}, \quad q_d = 1 - q_u.$$

Moreover $Y(i) = g(S(i))$, $i = 1,\dots,N+1$, is the intrinsic value of the American derivative. The following code defines a function *BinomialAmerican* which computes the binomial price and the cash flow of standard American derivatives.

```
function [P,C]=BinomialAmerican(Q,S,r,g)
h=Q(2)-Q(1);
syms x;
f = sym(g);
N=length(Q)-1;
expu=S(1,2)/S(1,1);
expd=S(2,2)/S(1,1);
qu=(exp(r*h)-expd)/(expu-expd);
qd=(expu-exp(r*h))/(expu-expd);
if (qu<0 || qd<0)
display('Error:  the market is not arbitrage free.');
P=0;
return
```

```
end
P=zeros(N+1);
P(:,N+1)=double(subs(f,x,S(:,N+1)));
C(:,N+1)=0;
Y=double(subs(f,x,S));
for j=N:-1:1
for i=1:j
P(i,j)=max(Y(i,j),exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1)));
C(i,j)=P(i,j)-exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1));
end
end
```

The binomial price of the derivative is stored in the upper-triangular $(N+1) \times (N+1)$ matrix $P$, while the cash flow is stored in the upper-triangular $(N+1) \times (N+1)$ matrix $C$. We use the convention that at time of maturity the cash flow is equal to zero.

For example, using as inputs the binomial stock price (2.20), the interest rate $r = 0.01$ and the pay-off function of the put with strike 10, i.e., $g(x) = (10 - x)+$, we obtain the output

$$P = \begin{array}{cccccc} 0.2385 & 0.1120 & 0.0320 & 0 & 0 & 0 \\ 0 & 0.3619 & 0.1899 & 0.0633 & 0 & 0 \\ 0 & 0 & 0.5297 & 0.3133 & 0.1250 & 0 \\ 0 & 0 & 0 & 0.7407 & 0.4970 & 0.2468 \\ 0 & 0 & 0 & 0 & 0.9752 & 0.7376 \\ 0 & 0 & 0 & 0 & 0 & 1.2038 \end{array}$$

$$C = \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0017 & 0.0017 & 0 \\ 0 & 0 & 0 & 0 & 0.0017 & 0 \end{array}$$

**Remark 4.2.** Using the pay-off of the call, $g(x) = (x - 10)_+$, we obtain that the price of the American call is exactly the same as the corresponding European call, see (3.12), while $C \equiv 0$. This of course is consistent with the proven fact that, in the absence of dividends, it is never optimal to exercise an American call prior to expire.

**Exercise 4.8.** *Repeat Exercise 3.9 for American put options.*

**Exercise 4.9.** *Plot the pairs $(t_i, S(t_i))$ in the plane, for each optimal exercise time $t_i$ of the American put. Experiment for different values of $K, T$. Use a large number of steps ($N \approx 100$). What happens when $T \to +\infty$?*

In the time-continuum limit, $N \to +\infty$, $h \to 0$ such that $Nh = T$, the points $(t_i, S(t_i))$ for each optimal exercise time form the so-called the **optimal exercise curve** (or **optimal exercise boundary**) of the American put option.

**Exercise 4.10.** *Show numerically that the initial binomial price of the American put with strike $K$ and maturity $T$ converges, as $T \to +\infty$, to the value $v(S_0)$, where $v$ is the function*

$$v(x) = \begin{cases} K - x & 0 \le x \le L \\ (K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & x > L \end{cases}$$

*and*

$$L = \frac{2r}{2r + \sigma^2} K.$$

*How is this result related to your findings in Exercise 4.9?*

# Chapter 5

# Introduction to Probability Theory

The main goal of this chapter is to reformulate the binomial options pricing model in the language of probability theory. To this purpose we shall first review some basic concept in probability; a more systematic presentation of this theory can be found e.g. in [5]. For an application of probability theory to portfolio optimization, see Appendix A.

## 5.1 Finite Probability Spaces

Let $\Omega$ be a set containing a finite number of elements $\omega_1, \omega_2, \dots, \omega_M$. We denote $\Omega$ as

$$\Omega = \{\omega_1, \dots, \omega_M\}, \quad \text{or} \quad \Omega = \{\omega_i\}_{i=1,\dots,M} \tag{5.1}$$

and call it a **sample space**. The elements $\omega_i \in \Omega$, $i = 1, \dots, M$, are called **sample points**. The sample points identify the possible outcomes of an experiment. For instance, if the experiment consists in "throwing a die", then

$$\Omega = \{1, 2, 3, 4, 5, 6\} \quad (M = 6),$$

while for the experiment "tossing a coin once", we have

$$\Omega = \Omega_1 := \{H, T\} \quad (M = 2),$$

where $H$ stands for "Head" and $T$ for "Tail"; the subscript 1 in $\Omega_1$ indicates that the coin is tossed only once. In the experiment "tossing a coin twice" we have

$$\Omega = \Omega_2 := \{(H, H), (H, T), (T, H), (T, T)\} = \Omega_1 \times \Omega_1 \quad (M = 2^2 = 4)$$

and in the experiment "tossing a coin $N$ times" we have

$$\begin{aligned} \Omega = \Omega_N &:= \{\omega = (\gamma_1, \gamma_2, \dots, \gamma_N);\ \gamma_j = H \text{ or } T,\ j = 1, \dots, N\} \\ &= \underbrace{\Omega_1 \times \Omega_1 \times \cdots \times \Omega_1}_{N \text{ times}} = \{H, T\}^N \quad (M = 2^N). \end{aligned}$$

We denote by $2^\Omega$ the **power set** of $\Omega$, i.e., the set of all subsets of $\Omega$. It consists of the empty set $\emptyset$, the subsets containing one element, i.e., $\{\omega_1\}, \{\omega_2\}, \dots, \{\omega_M\}$, which are called **atomic sets**, the subsets containing two elements, i.e.,

$$\{\omega_1, \omega_2\}, \dots, \{\omega_1, \omega_M\}, \{\omega_2, \omega_3\}, \dots, \{\omega_2, \omega_M\}, \dots, \{\omega_{M-1}, \omega_M\},$$

the subsets containing 3 elements and so on, and the set $\Omega = \{\omega_1, \dots, \omega_M\}$ itself. Thus $2^\Omega$ contains $2^M$ elements. For instance

$$2^{\Omega_1} = \{\emptyset, \{H\}, \{T\}, \{H, T\} = \Omega_1\}.$$

**Exercise 5.1.** *Write down* $2^{\Omega_2}$*.*

The elements of $2^\Omega$ (i.e., the subsets of $\Omega$) are called **events**. They identify possible events that occur in the experiment. For example

$$\{2, 4, 6\} \equiv [\text{the result of throwing a die is an even number}],$$

$$\{(H, H), (T, T)\} \equiv [\text{tossing a coin twice gives the same outcome in both tosses}].$$

**Exercise 5.2.** *Write down the following events* $A, B, C \in 2^{\Omega_4}$*:*

$$A = [\textit{number heads=number tails}], \quad B = [\textit{successive tosses are different}],$$

$$C = [\textit{there exist at least three identical tosses}].$$

If $A, B \in 2^\Omega$ are events, then $A \cup B$ is the event that $A$ *or* $B$ happens, while $A \cap B$ is the event that both $A$ *and* $B$ happen. If the sets $A, B \subset \Omega$ are **disjoint**, i.e., $A \cap B = \emptyset$, the events $A$ and $B$ cannot occur simultaneously. For instance, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$ means that the outcome of a die roll cannot be both even and odd.

The atomic set $\{\omega_i\}$ identifies the event that the outcome of the experiment is exactly $\omega_i$. We want to assign a probability $\mathbb{P}$ to such special events. To this purpose we introduce $M$ real numbers $p_1, p_2, \dots p_M$ such that

$$0 < p_i < 1, \text{ for all } i = 1, \dots, M, \quad \text{and} \quad \sum_{i=1}^{M} p_i = 1. \tag{5.2}$$

We define $p_i$ to be the probability of the event $\{\omega_i\}$, that is

$$\mathbb{P}(\{\omega_i\}) = p_i, \quad i = 1, \dots, M.$$

Any generic event $A \in 2^\Omega$ can be written as the disjoint union of atomic events, e.g.,

$$\{\omega_1, \omega_3, \omega_6\} = \{\omega_1\} \cup \{\omega_3\} \cup \{\omega_6\}.$$

This leads to define the probability of a generic event $A \in 2^\Omega$ as

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} \mathbb{P}(\{\omega_i\}) = \sum_{i:\omega_i \in A} p_i. \tag{5.3}$$

For instance $\mathbb{P}(\{\omega_1, \omega_3, \omega_6\}) = p_1 + p_3 + p_6$. We shall also write the definition of $\mathbb{P}(A)$ as

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

In particular

$$\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = \sum_{i=1}^{M} p_i = 1.$$

We also set

$$\mathbb{P}(\emptyset) = 0, \tag{5.4}$$

which means that it is impossible that the experiment gives no outcome. Clearly $\emptyset$ is the only event with zero probability: any other such event is excluded *a priori* by the sample space. At this point every event has been assigned a probability.

**Definition 5.1.** *Given $p = (p_1, \dots, p_M)$ satisfying* (5.2) *and a set $\Omega = \{\omega_1, \dots, \omega_M\}$, the function $\mathbb{P} : 2^\Omega \to [0, 1]$ defined by* (5.3)-(5.4) *is called a* **probability measure**. *The pair $(\Omega, \mathbb{P})$, is called a* **finite probability space**.

**Remark 5.1.** If we want to emphasize the dependence of the probability measure on the parameters $p_1, \dots, p_m$, we shall denote it by $\mathbb{P}_p$.

### Examples

- In the experiment "throwing a die", let $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$. We say that the die is **fair**, as any number between 1 and 6 has the same probability to appear as a result of the roll. Then $\mathbb{P}(\{2, 4, 6\}) = 1/6 + 1/6 + 1/6 = 1/2$, i.e., we have 50% chances to get an even number.

- In $\Omega_1$ we let $\mathbb{P}(\{H\}) = p_H$, $\mathbb{P}(\{T\}) = p_T$, where $p_T = 1 - p_H$, $p_H \in (0, 1)$. The coin is said to be **fair** if $p_H = p_T = 1/2$.

- In $\Omega_2$ we assign a probability to the atomic events by setting

  $$\mathbb{P}(\{(H, H)\}) = p_H \cdot p_H = p_H^2,$$
  $$\mathbb{P}(\{(T, H)\}) = \mathbb{P}(\{(H, T)\}) = p_T p_H, \quad \mathbb{P}(\{(T, T)\}) = p_T^2.$$

  Notice that $\mathbb{P}(\{(H, H)\}) + \mathbb{P}(\{(T, H)\}) + \mathbb{P}(\{(H, T)\}) + \mathbb{P}(\{(T, T)\}) = (p_H + p_T)^2 = 1$, as it should be. This way of assigning probabilities is equivalent to the assumption that the two tosses are **independent**, i.e., the result of the first toss does not influence the result of the second toss (see Definition 5.4 below). As an example of computing the probability of non-atomic events, consider the event

  $$A = [\textit{the outcome of the two tosses is the same}].$$

  Then

  $$\mathbb{P}(A) = \mathbb{P}(\{(H, H), (T, T)\}) = \mathbb{P}(\{(H, H)\}) + \mathbb{P}(\{(T, T)\}) = p_H^2 + p_T^2.$$

- In $\Omega_N = \{H,T\}^N$, we assign a probability to each of the $2^N$ atomic events by letting

$$\mathbb{P}(\{\omega\}) = (p_H)^{N_H(\omega)}(p_T)^{N_T(\omega)}, \quad \text{for all } \omega \in \Omega_N,$$

  where $N_H(\omega)$ and $N_T(\omega) = N - N_H(\omega)$ are respectively the number of heads and tails in the $N$-toss corresponding to $\omega$. Again, this definition of probability makes the $N$-tosses independent. Since for all $k = 0, \dots, N$ the number of $N$-tosses $\omega \in \Omega_N$ having $N_H(\omega) = k$ is given by the binomial coefficient

$$\binom{N}{k} = \frac{N!}{k!(N-k)!},$$

  then

$$\sum_{\omega\in\Omega_N} \mathbb{P}(\{\omega\}) = \sum_{\omega\in\Omega_N} (p_H)^{N_H(\omega)}(p_T)^{N_T(\omega)} = (p_T)^N \sum_{\omega\in\Omega_N} \left(\frac{p_H}{p_T}\right)^{N_H(\omega)}$$
$$= (p_T)^N \sum_{k=0}^{N} \binom{N}{k} \left(\frac{p_H}{p_T}\right)^k.$$

  By the binomial theorem, $(1+a)^N = \sum_{k=0}^N \binom{N}{k} a^k$, hence

$$\mathbb{P}(\Omega) = \sum_{\omega\in\Omega_N} \mathbb{P}(\{\omega\}) = (p_T)^N \left(1 + \frac{p_H}{p_T}\right)^N = (p_T + p_H)^N = 1.$$

  The probability of any other event $A \in 2^{\Omega_N}$ is the sum of the probabilities of the atomic events whose (disjoint) union forms the set $A$.

The last example deserves to be given a separate definition.

**Definition 5.2.** *Given* $0 < p < 1$*, the pair* $(\Omega_N, \mathbb{P}_p)$ *given by* $\Omega_N = \{H,T\}^N$ *and*

$$\mathbb{P}_p(A) = \sum_{\omega\in A} p^{N_H(\omega)}(1-p)^{N_T(\omega)}, \quad \text{for all } A \in 2^{\Omega_N}, \tag{5.5}$$

*is called a* $N$**-coin toss probability space***. Here* $N_H(\omega)$ *is the number of* $H$ *in the sample* $\omega$ *and* $N_T(\omega) = N - N_H(\omega)$ *is the number of* $T$*.*

**Example.** Given $k \in \{0,\dots,N\}$, let $A_k = \{\omega : N_H(\omega) = k\} \subset \Omega_N$ be the event that the number of heads in the $N$-toss is $k$. Then

$$\mathbb{P}(A_k) = \sum_{\omega\in A_k} p^{N_H(\omega)}(1-p)^{N_T(\omega)} = \sum_{\omega\in A_k} p^k(1-p)^{N-k} = p^k(1-p)^{N-k} \sum_{\omega\in A_k} 1 = p^k(1-p)^{N-k} M_k,$$

where $M_k = \sum_{\omega\in A_k} 1$ is the number of $N$-tosses with $k$ heads. As $M_k = \binom{N}{k}$ we get

$$\mathbb{P}(A_k) = \binom{N}{k} p^k (1-p)^{N-k}.$$

**Exercise 5.3.** *Compute* $\mathbb{P}(A)$*, where* $A = \{\omega : N_H(\omega) = N_T(\omega)\} \subset \Omega_N$*.*

It is possible that the occurrence of an event $A$ affects the probability that a second event $B$ occurred. For instance, for a fair coin we have $\mathbb{P}(\{H, H\}) = 1/4$, but if we know that the first toss is a tail, then $\mathbb{P}(\{H, H\}) = 0$. This simple remark leads to the definition of conditional probability.

**Definition 5.3.** *Given two events* $A, B$ *such that* $\mathbb{P}(B) > 0$*, the* **conditional probability** *of* $A$ *given* $B$ *is defined as*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

*Similarly, if* $B_1, B_2, \ldots, B_n$ *are events such that* $\mathbb{P}(B_1 \cap \cdots \cap B_n) > 0$*, the conditional probability of* $A$ *given*[1] $B_1, \ldots, B_n$ *is*

$$\mathbb{P}(A|B_1, \ldots, B_n) = \frac{\mathbb{P}(A \cap B_1 \cap \cdots \cap B_n)}{\mathbb{P}(B_1 \cap \cdots \cap B_n)}.$$

If the occurrence of $B$ does not affect the occurrence of $A$, i.e., if $\mathbb{P}(A|B) = \mathbb{P}(A)$, we say that the two events are independent. By the previous definition, the independence property is equivalent to the following.

**Definition 5.4.** *Two events* $A, B$ *are said to be* **independent** *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

*Similarly,* $n$ *events* $A_1, \ldots, A_n$ *are said to be independent if*

$$\mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_m}) = \mathbb{P}(A_{k_1}) \cdot \ldots \cdot \mathbb{P}(A_{k_m}),$$

*for all* $1 \leq k_1 < k_2 < \cdots < k_m \leq n$*.*

**Exercise 5.4.** *Write down all the identities that must be verified in order for three events* $A, B, C$ *to be independent.*

**Exercise 5.5.** *Compute the probability of the events defined in Exercise 5.2 and verify that there is no pair of independent events among them. Compute*

$$\mathbb{P}(A|B), \quad \mathbb{P}(B|A), \quad \mathbb{P}(A|C), \quad \mathbb{P}(B|C).$$

*Give an example of two non-disjoint independent events defined on* $\Omega_4$*.*

[1] i.e., given the simultaneous occurrence of the events $B_1, \ldots, B_n$.

## 5.2 Random Variables

In general the purpose of an experiment is to determine the value of quantities which depend on the outcome of the experiment (e.g., the velocity of a particle, which is determined by successive measurements of its position). We call such quantities random variables.

**Definition 5.5.** *Let $(\Omega, \mathbb{P})$ be a finite probability space. A* **random variable** *is a function $X : \Omega \to \mathbb{R}$. If $g : \mathbb{R} \to \mathbb{R}$, then the random variable $Y = g(X)$ is said to be $X$-measurable.*

Note that the property of $Y$ being $X$-measurable means that the value attained by $Y$ can be inferred by the value attained by $X$, i.e., $Y(\omega) = g(X(\omega))$.

Since $\Omega = \{\omega_1, \dots, \omega_M\}$, then $X$ can attain only a finite number of values, which we denote $x_1, \dots, x_M$, namely

$$X(\omega_i) = x_i, \quad i = 1, \dots, M.$$

The values $x_1, \dots, x_M$ need *not* be distinct. If $X(\omega_i) = c$, for all $i = 1, \dots, M$, we say that $X$ is a non-random, or **deterministic**, constant (the value of $X$ is independent of the outcome of the experiment).

The **image** of $X$ is the finite set defined as

$$\mathrm{Im}(X) = \{x \in \mathbb{R} \text{ such that } X(\omega) = x, \text{ for some } \omega \in \Omega\},$$

that is the set of possible values attainable by $X$. Given $a \in \mathbb{R}$, we denote

$$\{X = a\} = \{\omega \in \Omega : X(\omega) = a\},$$

which is the event that $X$ attains the value $a$. Of course, $\{X = a\} = \emptyset$ if $a \notin \mathrm{Im}(X)$. In general, given $I \subseteq \mathbb{R}$, we denote

$$\{X \in I\} = \{\omega \in \Omega : X(\omega) \in I\},$$

which is the event that the value attained by $X$ lies in the set $I$. Moreover we denote

$$\{X = a, Y = b\} = \{X = a\} \cap \{Y = b\}, \quad \{X \in I_1, Y \in I_2\} = \{X \in I_1\} \cap \{Y \in I_2\}.$$

The probability that $X$ takes value $a$ is given by

$$\mathbb{P}(X = a) = \mathbb{P}(\{X = a\}) = \sum_{i : X(\omega_i) = a} p_i.$$

If $a \notin \mathrm{Im}(X)$, then $\mathbb{P}(X = a) = \mathbb{P}(\emptyset) = 0$. More generally, given any open subset $I$ of $\mathbb{R}$, we write

$$\mathbb{P}(X \in I) = \mathbb{P}(\{X \in I\}) = \sum_{i : X(\omega_i) \in I} p_i,$$

which is the probability that the value of $X$ belongs to $I$. For example, in the probability space of a fair die consider the random variable

$$X(\omega) = (-1)^{\omega}, \quad \omega \in \{1,2,3,4,5,6\}. \tag{5.6}$$

Then $X(\omega) = 1$ if $\omega$ is even and $X(\omega) = -1$ is $\omega$ is odd. Moreover

$$\mathbb{P}(X = 1) = \mathbb{P}(\{2,4,6\}) = 1/2, \quad \mathbb{P}(X = -1) = \mathbb{P}(\{1,3,5\}) = 1/2,$$

whereas

$$\mathbb{P}(X \neq \pm 1) = \mathbb{P}(\emptyset) = 0.$$

The set $A = \{2,4,6\}$ is said to be **resolved** by $X$, because the occurrence of the event $A$ (i.e., the fact that the outcome of the throw is an even number) is equivalent to $X$ taking value 1. In general, given a random variable $X : \Omega \to \mathbb{R}$, the events resolved by $X$ are the sets of the form $\{X \in I\}$, for some $I \subseteq \mathbb{R}$. These events comprise the so called **information carried by** $X$. The idea is that even if the outcome of an experiment is unknown, measuring the value attained by a random variable gives some information on the result of the experiment.

**Definition 5.6.** *Given a random variable $X : \Omega \to \mathbb{R}$, the function $f_X : \mathbb{R} \to [0,1]$ defined by*

$$f_X(x) = \mathbb{P}(X = x)$$

*is called the* **distribution** *of $X$, while $F_X : \mathbb{R} \to [0,1]$ given by*

$$F_X(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}$$

*is called the* **cumulative distribution** *of $X$.*

Note that $f_X(x)$ is non-zero if only if $x \in \mathrm{Im}(X)$, and that $F_X$ is a non-decreasing function. For example, for the random variable (5.6) defined on the probability space of a fair die we have

$$F_X(x) = \begin{cases} 0, & x < -1, \\ 1/2, & x \in [-1,1), \\ 1, & x \geq 1. \end{cases}$$

The probability that a random variable $X$ takes value in the interval $[a,b]$ can be written in terms of the distribution of $X$ as

$$\mathbb{P}(a \leq X \leq b) = \sum_{i : X(\omega_i) = x_i \in [a,b]} \mathbb{P}(X = x_i) = \sum_{i : a \leq x_i \leq b} f_X(x_i). \tag{5.7}$$

In a similar fashion, if $g : \mathbb{R} \to \mathbb{R}$ then

$$\mathbb{P}(a \leq g(X) \leq b) = \sum_{i : g(X(\omega_i)) = g(x_i) \in [a,b]} \mathbb{P}(X = x_i) = \sum_{i : a \leq g(x_i) \leq b} f_X(x_i). \tag{5.8}$$

We shall use these formulas later on.

### 5.2.1 Expectation and Variance

Next we define the expectation and variance of random variables. We may think of the expectation of $X$ as an estimate on the average value of $X$ and the variance of $X$ as a measure of how far is this estimate from to the precise value of $X$.

**Definition 5.7.** *Given a finite probability space* $(\Omega, \mathbb{P})$*, the* **expectation** *(or* **expected value***) of* $X : \Omega \to \mathbb{R}$ *is defined by*

$$\mathbb{E}[X] = \sum_{i=1}^{M} X(\omega_i)\mathbb{P}(\omega_i).$$

We shall also write the definition of $\mathbb{E}[X]$ as

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\{\omega\}). \tag{5.9}$$

For instance, in the $N$-coin toss probability space $(\Omega_N, \mathbb{P}_p)$ we have

$$\mathbb{E}[X] = \sum_{\omega \in \Omega_N} X(\omega)p^{N_H(\omega)}(1-p)^{N_T(\omega)}, \tag{5.10}$$

where $N_H(\omega)$ is the number of heads and $N_T(\omega) = N - N_H(\omega)$ is the number of tails in the $N$-toss $\omega \in \Omega_N$, see Definition 5.2.

**Exercise 5.6** ($\star$)**.** *Let* $X : \Omega_N \to \mathbb{R}$*,* $X(\omega) = N_H(\omega) - N_T(\omega)$*. Compute* $\mathbb{E}[X]$*. HINT: Use the identity* $\binom{N}{k}k = \binom{N-1}{k-1}N$*.*

We can rewrite the definition of expectation as

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}(X)} x\, \mathbb{P}(X = x),$$

or equivalently,

$$\mathbb{E}[X] = \sum_{x \in \mathrm{Im}(X)} x f_X(x). \tag{5.11}$$

The importance of (5.11) is that it allows to compute the expectation of $X$ from its distribution, without any reference to the original probability space. For instance, if we are told that a random variable $X$ takes the following values:

$$X = \begin{cases} 1 & \text{with probability } 1/4 \\ 2 & \text{with probability } 1/4 \\ -1 & \text{with probability } 1/2 \end{cases}, \tag{5.12}$$

then we can compute $\mathbb{E}[X]$ using (5.11) as

$$\mathbb{E}[X] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} - 1 \cdot \frac{1}{2} = \frac{1}{4}.$$

Some simple properties of the expectation are collected in the following theorem.

**Theorem 5.1.** *Let $X, Y$ be random variables on a finite probability space $(\Omega, \mathbb{P})$, $g : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$. The following holds:*

1. $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ *(linearity).*
2. *If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.*
3. *If $Y = g(X)$, then*
$$\mathbb{E}[g(X)] = \sum_{x \in \mathrm{Im}(X)} g(x) f_X(x). \tag{5.13}$$

**Exercise 5.7.** *Prove Theorem 5.1.*

**Definition 5.8.** *The* **variance** *of a random variable $X : \Omega \to \mathbb{R}$ is defined by*

$$\mathrm{Var}[X] = \mathbb{E}[(\mathbb{E}[X] - X)^2].$$

Using the linearity of the expectation, we obtain easily the formula

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{5.14}$$

Moreover $\mathrm{Var}[aX] = a^2 \mathrm{Var}[X]$ holds for all constants $a \in \mathbb{R}$. The variance of the sum of two random variables is in general different from the sum of their variances, unless the random variables are uncorrelated (see Theorem 5.2 below). Furthermore the variance of a random variable is always non-negative and it is zero if and only if the random variable is a deterministic constant. Hence we may also interpret the variance as a measure of the "randomness" of a random variable.

Using (5.13) with $g(x) = x^2$, we can rewrite the definition of variance in terms of the distribution function of $X$ as

$$\mathrm{Var}[X] = \sum_{x \in \mathrm{Im}(X)} x^2 f_X(x) - \left( \sum_{x \in \mathrm{Im}(X)} x f_X(x) \right)^2, \tag{5.15}$$

which allows to compute $\mathrm{Var}[X]$ without any reference to the original probability space. For instance for the random variable (5.12) we find

$$\mathrm{Var}[X] = 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} - \left(\frac{1}{4}\right)^2 = \frac{27}{16}.$$

**Exercise 5.8** ($\star$)**.** *Let $R$ be the relative return of a portfolio which is long 1 share of the derivative in Section 3.1.2. Compute $\mathbb{E}[R]$ and $\mathrm{Var}[R]$.*

**Remark 5.2.** The variance of the relative return of a portfolio is a measure of the portfolio **risk**, see Appendix A.

**Example: mean of log return and volatility of the binomial stock price**

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with $t_i - t_{i-1} = h$, for all $i = 1, \dots N$. Given $u > d$, consider a random variable $X$ such that $X = u$ with probability $p$ and $X = d$ with probability $1 - p$. We may think of $X$ as being defined on $\Omega_1 = \{H, T\}$, with $X(H) = u$ and $X(T) = d$. Now, the binomial stock price at time $t_i$ can be written as $S(t_i) = S(t_{i-1}) \exp(X)$. Hence the log-return of the stock in the interval $[t_{i-1}, t_i]$ is

$$R = \log S(t_i) - \log S(t_{i-1}) = \log \frac{S(t_i)}{S(t_{i-1})} = X.$$

It follows that the expectation and the variance of the log-return of the stock in the interval $[t_{i-1}, t_i]$ is

$$\mathbb{E}[R] = \mathbb{E}[X] = (pu + (1 - p)d),$$

$$\mathrm{Var}[R] = \mathrm{Var}[X] = [pu^2 + (1 - p)d^2 - (pu + (1 - p)d)^2)] = p(1 - p)(u - d)^2.$$

Thus the parameters $\alpha, \sigma^2$ defined in (2.19) can be rewritten as

$$\alpha = \frac{1}{h}\mathbb{E}[\log S(t_i) - \log S(t_{i-1})], \quad \sigma^2 = \frac{1}{h}\mathrm{Var}[\log S(t_i) - \log S(t_{i-1})]. \tag{5.16}$$

We see now that $\alpha$ is the expected log-return of the binomial stock price in the interval $[t_{i-1}, t_i]$ per unit of time (instantaneous log-return), while $\sigma^2$ is the instantaneous variance. The parameter $\sigma$ itself is called instantaneous volatility of the binomial stock. It is part of the assumptions in the binomial model that the parameters $\alpha$ and $\sigma$ are the same for every interval $[t_{i-1}, t_i]$ of the partition.

### 5.2.2 Independence and Correlation

We have seen before that a random variable $X$ carries information. Now, if $Y = g(X)$ for some (non-constant) function $g : \mathbb{R} \to \mathbb{R}$, then $Y$ carries no more information than $X$: any event resolved by knowing the value of $Y$ is also resolved by knowing the value of $X$. The other extreme case is when two random variables carry independent information.

**Definition 5.9.** *Two random variables $X_1, X_2 : \Omega \to \mathbb{R}$ are said to be* **independent** *if the events $\{X_1 \in I_1\}, \{X_2 \in I_2\}$ are independent events, for all regular[2] sets $I_1, I_2 \subset \mathbb{R}$. This means that*

$$\mathbb{P}(X \in I_1, X_2 \in I_2) = \mathbb{P}(X_1 \in I_1)\mathbb{P}(X_2 \in I_2).$$

*More generally, $n$ random variables $X_1, \dots, X_n : \Omega \to \mathbb{R}$ are independent if the events $\{X_1 \in I_1\}, \{X_2 \in I_2\}, \dots, \{X_n \in I_n\}$ are independent (in the sense of Definition 5.4 ) for all regular sets $I_1, I_2, \dots I_n \subset \mathbb{R}$.*

[2]By regular sets we mean intervals or sets which can be written as the union of countably many intervals.

**Exercise 5.9** (•). *Show that when $X, Y$ are independent random variables, then the only events which are resolved by both variables are $\emptyset$ and $\Omega$. Show that two deterministic constants are always independent. Finally assume $Y = g(X)$ and show that in this case the two random variables are independent if and only if $Y$ is a deterministic constant.*

Note that the independence property is connected with the probability defined on the sample space. Thus two random variables may be independent with respect to some probability and not-independent with respect to another. We shall use later the following important result:

**Theorem 5.2.** *Let $X_1, X_2, \dots X_n$ be independent random variables, $k \in \{1, \dots, n-1\}$ and $g : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R}^{n-k} \to \mathbb{R}$. Then the random variables*

$$Y = g(X_1, X_2, \dots, X_k), \quad Z = f(X_{k+1}, \cdots, X_n)$$

*are independent.*

**Exercise 5.10** (•). *Prove the theorem for $n = 2$.*

**Definition 5.10.** *The* **covariance** *of two random variables $X, Y : \Omega \to \mathbb{R}$ is defined by*

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(\mathbb{E}[X] - X)(\mathbb{E}[Y] - Y)].$$

*If $\mathrm{Cov}(X, Y) = 0$, the two random variables are said to be* **uncorrelated**.

Using the linearity of the expectation we can rewrite the definition of covariance as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]. \tag{5.17}$$

The interpretation is the following: If $\mathrm{Cov}(X, Y) > 0$, then $Y$ tends to increase (resp. decrease) when $X$ increases (resp. decrease), while if $\mathrm{Cov}(X, Y) < 0$, the two variables have tendency to move in the opposition direction. For instance, assuming $\mathrm{Var}[X] > 0$,

$$\mathrm{Cov}(X, 2X) = 2\mathrm{Var}[X] > 0, \quad \mathrm{Cov}(X, -2X) = -2\mathrm{Var}[X] < 0.$$

**Exercise 5.11** (•). *Let $(\Omega, \mathbb{P})$ be a finite probability space and $X, Y : \Omega \to \mathbb{R}$ be two random variables. Prove that $X, Y$ independent $\Rightarrow$ $X, Y$ uncorrelated. Show with a counterexample that the opposite implication is not true. Finally show that*

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \quad \textit{if and only if } X, Y \textit{ are uncorrelated.} \tag{5.18}$$

**Exercise 5.12** (•). *Assume that $X, Y$ are not deterministic constants. Prove the inequality*

$$-\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]} \le \mathrm{Cov}(X, Y) \le \sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}. \tag{5.19}$$

*Show that the left (resp. right) inequality becomes an equality if and only if there exists a negative (resp. positive) constant $a_0$ and a real constant $b_0$ such that $Y = a_0 X + b_0$.*

By inequality (5.19) it is convenient to introduce the following definition.

**Definition 5.11.** *Let $X, Y : \Omega \to \mathbb{R}$ be two random variables such that* $\mathrm{Var}[X]$ *and* $\mathrm{Var}[Y]$ *are both positive (i.e., $X, Y$ are not deterministic constants). Then*

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \in [-1,1]$$

*is called the* **correlation** *of $X, Y$.*

Hence, the closer is $\mathrm{Cor}(X,Y)$ to 1 (resp. $-1$) the more $Y$ has the tendency to move in the same (resp. opposite) direction of $X$. For instance $\mathrm{Cor}(X,2X) = 1$, and $\mathrm{Cor}(X,-2X) = -1$.

**Exercise 5.13.** *Compute the correlation of the random variables $X, Y : \Omega_3 \to \mathbb{R}$ given by $X(\omega) = N_T(\omega) - N_H(\omega)$, $Y(\omega) = N_H(\omega)$.*

**Definition 5.12.** *If $X, Y$ are two random variables on a finite probability space, then the function $f_{X,Y} : \mathbb{R}^2 \to [0,1]$ given by*

$$f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

*is called the* **joint distribution** *of $X$ and $Y$.*

**Exercise 5.14.** *Let $X, Y$ have the joint distribution $f_{X,Y}$. Show that the distributions of $X$ and $Y$ are given by*

$$f_X(x) = \sum_{y \in \mathrm{Im}(Y)} f_{X,Y}(x,y), \quad f_Y(y) = \sum_{x \in \mathrm{Im}(X)} f_{X,Y}(x,y)$$

*Show that $X, Y$ are independent if and only if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.*

If the joint distribution is given, then the covariance of two random variables can easily be computed without any reference to the original probability space. In fact, since

$$\mathbb{E}[XY] = \sum_{i=1}^{M} X(\omega_i)Y(\omega_i)\mathbb{P}(\omega_i) = \sum_{x \in \mathrm{Im}(X)} \sum_{y \in \mathrm{Im}(Y)} x\, y\, f_{X,Y}(x,y), \tag{5.20}$$

then

$$\mathrm{Cov}(X,Y) = \sum_{x \in \mathrm{Im}(X)} \sum_{y \in \mathrm{Im}(Y)} x\, y\, f_{X,Y}(x,y) - \sum_{x \in \mathrm{Im}(X)} x f_X(x) \sum_{y \in \mathrm{Im}(Y)} y\, f_Y(y), \tag{5.21}$$

where the distributions $f_X, f_Y$ are computed from $f_{X,Y}$ as shown in Exercise 5.14. In conclusion, if the joint distribution of two random variables $X, Y$ is given, then all relevant information on $X, Y$ (independence, expectation, variance, correlation, etc.) can be inferred without any reference to the original probability space.

**Example.** Consider two random variables $X, Y$ such that $\mathrm{Im}(X) = \{-1, 1, 3, 4\}$, $\mathrm{Im}(Y) = \{-1, 0, 1, 2\}$ and let their joint probability distribution be defined as in the following table

| $Y\downarrow$, $X\rightarrow$ | -1 | 1 | 3 | 4 |
|---|---|---|---|---|
| -1 | 1/64 | 2/64 | 1/64 | 4/64 |
| 0 | 5/64 | 1/64 | 9/64 | 6/64 |
| 1 | 6/64 | 10/64 | 1/64 | 12/64 |
| 2 | 2/64 | 1/64 | 1/64 | 2/64 |

For instance, $f_{X,Y}(-1,1) = 1/64$, $f_{X,Y}(-1,0) = 5/64$, and so on. Let us compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathrm{Cov}(X,Y)$.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{x\in \mathrm{Im}(X)} x f_X(x) = \sum_{x\in \mathrm{Im}(X)} x \sum_{y\in \mathrm{Im}(Y)} f_{X,Y}(x,y)\\ &= -\sum_{y\in \mathrm{Im}(Y)} f_{X,Y}(-1,y) + \sum_{y\in \mathrm{Im}(Y)} f_{X,Y}(1,y) + 3\sum_{y\in \mathrm{Im}(Y)} f_{X,Y}(3,y) + 4\sum_{y\in \mathrm{Im}(Y)} f_{X,Y}(4,y)\\ &= -(\frac{1}{64}+\frac{5}{64}+\frac{6}{64}+\frac{2}{64}) + (\frac{2}{64}+\frac{1}{64}+\frac{10}{64}+\frac{1}{64})\\ &\quad + 3(\frac{1}{64}+\frac{9}{64}+\frac{1}{64}+\frac{1}{64}) + 4(\frac{4}{64}+\frac{6}{64}+\frac{12}{64}+\frac{2}{64}) = \frac{33}{16}.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{y\in \mathrm{Im}(Y)} y f_Y(y) = \sum_{y\in \mathrm{Im}(Y)} y \sum_{x\in \mathrm{Im}(X)} f_{X,Y}(x,y)\\ &= -\sum_{x\in \mathrm{Im}(X)} f_{X,Y}(x,-1) + 0\cdot\sum_{x\in \mathrm{Im}(X)} f_{X,Y}(x,0) + \sum_{x\in \mathrm{Im}(X)} f_{X,Y}(x,1) + 2\sum_{x\in \mathrm{Im}(X)} f_{X,Y}(x,2)\\ &= -(\frac{1}{64}+\frac{2}{64}+\frac{1}{64}+\frac{4}{64}) + 0\cdot(\frac{5}{64}+\frac{1}{64}+\frac{9}{64}+\frac{6}{64})\\ &\quad + (\frac{6}{64}+\frac{10}{64}+\frac{1}{64}+\frac{12}{64}) + 2(\frac{2}{64}+\frac{1}{64}+\frac{1}{64}+\frac{2}{64}) = \frac{33}{64}.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{x\in \mathrm{Im}(X)} \sum_{y\in \mathrm{Im}(Y)} x\, y\, f_{X,Y}(x,y)\\ &= (-1)(-1)\frac{1}{64} + 1(-1)\frac{2}{64} + 3(-1)\frac{1}{64}\\ &\quad + \cdots\cdots\cdots\cdots = \frac{55}{64}.\end{aligned}$$

Hence $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \approx -0.204$.

**Exercise 5.15.** *Compute* $\mathrm{Cor}(X,Y)$.

### 5.2.3 Conditional expectation

If $X, Y$ are independent random variables, knowing the value of $Y$ does not help to estimate the random variable $X$. However if $X, Y$ are not independent, then we may use the information on the value attained by $Y$ to find an estimate of $X$ which is better than $\mathbb{E}[X]$. This leads to the important concept of **conditional expectation**.

**Definition 5.13.** *Let $(\Omega, \mathbb{P})$ be a finite probability space, $X, Y : \Omega \to \mathbb{R}$ random variables and $y \in \mathbb{R}$. The expectation of $X$ conditional to $Y = y$ (or given the event $\{Y = y\}$) is defined as*

$$\mathbb{E}[X|Y = y] = \sum_{x \in \mathrm{Im(X)}} \mathbb{P}(X = x|Y = y)\, x$$

*where $\mathbb{P}(X = x|Y = y)$ is the conditional probability of the event $\{X = x\}$, given the event $\{Y = y\}$ (see Def. 5.3). The random variable*

$$\mathbb{E}[X|Y] : \Omega \to \mathbb{R}, \quad \mathbb{E}[X|Y](\omega) = \mathbb{E}[X|Y = Y(\omega)]$$

*is called the expectation of $X$ conditional to $Y$.*

In a similar fashion one defines the conditional expectation with respect to several random variables, i.e., $\mathbb{E}[X|Y_1 = y_1, Y_2 = y_2, \dots Y_N = y_N]$ and $\mathbb{E}[X|Y_1, \dots, Y_N]$. For example, in the probability space of a fair die, consider

$$X(\omega) = (-1)^\omega, \quad Y(\omega) = (\omega - 1)(\omega - 2)(\omega - 3), \quad \omega \in \{1, 2, 3, 4, 5, 6\}. \tag{5.22}$$

Note that $\mathrm{Im}(Y) = \{0, 6, 24, 60\}$. Then we compute

$$\begin{aligned}
\mathbb{E}[X|Y = 0] &= \mathbb{P}(X = 1|Y = 0) - \mathbb{P}(X = -1|Y = 0) \\
&= \frac{\mathbb{P}(X = 1, Y = 0)}{\mathbb{P}(Y = 0)} - \frac{\mathbb{P}(X = -1, Y = 0)}{\mathbb{P}(Y = 0)} \\
&= \frac{\mathbb{P}(\{2\})}{\mathbb{P}(\{1, 2, 3\})} - \frac{\mathbb{P}(\{1, 3\})}{\mathbb{P}(\{1, 2, 3\})} = -1/3.
\end{aligned}$$

Similarly we find

$$\mathbb{E}[X|Y = 6] = 1, \quad \mathbb{E}[X|Y = 24] = -1, \quad \mathbb{E}[X|Y = 60] = 1,$$

hence $\mathbb{E}[X|Y]$ is the random variable

$$\mathbb{E}[X|Y](\omega) = \begin{cases} -1/3 & \text{if } \omega = 1, 2 \text{ or } 3 \\ 1 & \text{if } \omega = 4 \text{ or } 6 \\ -1 & \text{if } \omega = 5. \end{cases}$$

The following theorem collects a few important properties of the conditional expectation that will be used later on.

**Theorem 5.3.** *Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables on the finite probability space $(\Omega, \mathbb{P})$. Then*

1. *The conditional expectation is a linear operator, i.e.,*

$$\mathbb{E}[\alpha X + \beta Y|Z] = \alpha \mathbb{E}[X|Z] + \beta \mathbb{E}[Y|Z],$$

   *for all $\alpha, \beta \in \mathbb{R}$;*

*2. If $X$ is independent of $Y$, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$;*

*3. If $X$ is measurable with respect to $Y$, i.e., $X = g(Y)$ for some function $g$, then $\mathbb{E}[X|Y] = X$;*

*4. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$;*

*5. If $X$ is measurable with respect to $Z$, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$;*

*6. If $Z$ is measurable with respect to $Y$ then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$.*

*These properties remain true if the conditional expectation is taken with respect to several random variables.*

The interpretation of 2-3 in the previous theorem is the following: If $X$ is independent of $Y$, then the information carried by $Y$ does not help to improve our estimate on $X$ and thus our best estimate for $X$ remains $\mathbb{E}[X]$. On the other hand, if $X$ is measurable with respect of $Y$, then by knowing $Y$ we also know $X$ and thus our best estimate on $X$ is $X$ itself.

**Exercise 5.16** (?). *What is the interpretation of 4, 5, 6 in Theorem 5.3?*

## 5.3 Stochastic processes. Martingales

**Definition 5.14.** *Let $T > 0$. A one parameter family of random variables, $X(t) : \Omega \to \mathbb{R}$, $t \in [0, T]$, is called a* **stochastic process***. We denote a stochastic process by $\{X(t)\}_{t\in[0,T]}$ and by $X(t, \omega)$ the value of the random variable $X(t)$ on the sample $\omega \in \Omega$. For each fixed $\omega \in \Omega$, the curve $t \to X(t, \omega)$, is called a* **path** *of the stochastic process.*

We refer to the parameter $t$ as the time variable. If $X(t, \omega) = C(t)$, for all $\omega \in \Omega$, i.e., if the paths are the same for all sample points, we say that the stochastic process is a non-random (or **deterministic**) function of time. If $t$ runs over a (possibly infinite) discrete set $\{t_1, t_2, \dots\} \subset [0, T]$, then we say that the stochastic process is **discrete**. Note that a discrete stochastic process is equivalent to a sequence of random variables:

$$X = \{X_1, X_2, \dots\} \text{ where } X_i = X(t_i),\ i = 1, 2, \dots.$$

As a way of example, consider the following (discrete) stochastic process defined on the $N$-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$$\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N, \quad X_i(\omega) = \begin{cases} 1 & \text{if } \gamma_i = H \\ -1 & \text{if } \gamma_i = T \end{cases}$$

Clearly, the random variables $X_1, \dots, X_N$ are independent and have all the same distribution. In particular,

$$\mathbb{P}_p(X_i = 1) = p, \quad \mathbb{P}_p(X_i = -1) = 1 - p, \quad \text{for all } i = 1, \dots, N.$$

From now on we assume that the coin is fair ($p = 1/2$); we then have

$$\mathbb{E}[X_i] = 0, \quad \mathrm{Var}[X_i] = 1, \quad \text{for all } i = 1, \dots, N.$$

Moreover, for $n = 0, \dots, N$ we set

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i, \quad \text{for } n \geq 1.$$

The stochastic process $(M_0, \dots, M_N)$ is called **symmetric random walk**. It is centered in zero, which means that

$$\mathbb{E}[M_n] = 0, \quad \text{for all } n = 0, \dots, N.$$

Moreover, since it is the sum of independent random variables, the symmetric random walk has variance given by

$$\mathrm{Var}(M_n) = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = n,$$

see Theorem 5.2.

To understand the meaning of the term "random walk", consider a particle moving on the real line in the following way: if $X_i = 1$ (i.e., if the $i^{\text{th}}$ toss is a head), at time $t = i$ the particle moves one unit of length to the right, while if $X_i = -1$ (i.e., if the $i^{\text{th}}$ toss is a tail) it moves one unit of length to the left. Hence $M_n$ gives the total amount of units of length that the particle has traveled to the right or to the left up to the time $n$. The **increments** of the random walk are defined as follows. Given $0 = k_0 < k_1 < \cdots < k_m = N$, we let

$$\Delta_i = M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j, \quad i = 0, \dots, m-1,$$

i.e., $\Delta_0 = M_{k_1}$, $\Delta_1 = M_{k_2} - M_{k_1}, \dots, \Delta_{m-1} = M_{k_m} - M_{k_{m-1}}$. Hence $\Delta_j$ is the total displacement of the particle from time $k_{j-1}$ to time $k_j$. It follows by Theorem 5.2 that the increments of the random walk are independent random variables, that is to say, the distance traveled by the particle in the time interval $[k_{j-1}, k_j]$ is independent of the movements during any earlier or later time interval. Moreover

$$\mathbb{E}[\Delta_i] = 0, \quad \mathrm{Var}[\Delta_i] = k_{i+1} - k_i. \tag{5.23}$$

**Exercise 5.17** (•). *Let $T > 0$ and $n \in \mathbb{N}$ be given. Define the stochastic process*

$$\{W_n(t)\}_{t\in[0,T]}, \quad W_n(t) = \frac{1}{\sqrt{n}} M_{[nt]}, \tag{5.24}$$

*where $[z]$ denotes the greatest integer smaller than or equal to $z$ and $M_k = X_1 + X_2 + \cdots + X_k$, $k = 1, \dots, N$, is a symmetric random walk. It is assumed that the stochastic process $(X_1, \dots, X_N)$ is defined for $N > [nT]$, so that $W_n(t)$ is defined for all $t \in [0, T]$. Compute* $\mathbb{E}[W_n(t)]$, $\mathrm{Var}[W_n(t)]$, $\mathrm{Cov}[W_n(t), W_n(s)]$. *Show that* $\mathrm{Var}(W_n(t)) \to t$ *and* $\mathrm{Cov}(W_n(t), W_n(s)) \to \min(s, t)$ *as $n \to +\infty$.*

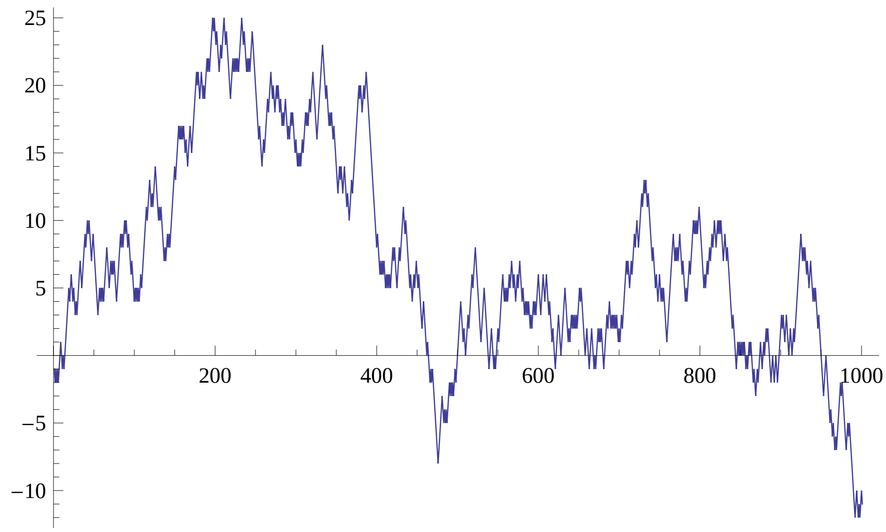



Figure 5.1: A path of the stochastic process (5.24) for $n = 1000$.

**Remark 5.3.** For large $n \in \mathbb{N}$, the process $\{W_n(t)\}_{t\in[0,T]}$ can be used an approximation for the Brownian motion, see Definition 5.18 below. An example of path of the stochastic process $\{W_n(t)\}_{t\in[0,T]}$ for $n = 1000$ is shown in Figure 5.1.

**Exercise 5.18 (Matlab).** *Write a Matlab function that generates a random path of the stochastic process* $\{W_n(t)\}_{t\in[0,T]}$.

To conclude this section we introduce the fundamental concept of martingale. Roughly speaking, a martingale is a stochastic process which has no tendency to rise or fall. The precise definition is the following.

**Definition 5.15.** *A discrete stochastic process* $\{X_1, X_2, \dots\}$ *on the finite probability space* $(\Omega, \mathbb{P})$ *is called a* **martingale** *if*

$$\mathbb{E}[X_{i+1}|X_1, X_2, \dots X_i] = X_i, \quad \textit{for all } i \geq 1. \tag{5.25}$$

The interpretation is the following: The variables $X_1, X_2, \dots X_i$ contains the information obtained by "looking" at the stochastic process up to the step $i$. For a martingale process, this information is not enough to infer whether, in the next step, the process will raise or fall. Note carefully that the martingale property depends on the probability being used: if $\{X_1, X_2, \dots\}$ is a martingale in the probability $\mathbb{P}$ and $\mathbb{P}'$ is another probability measure on the sample space $\Omega$, then $\{X_1, X_2, \dots\}$ need not be a martingale with respect to $\mathbb{P}'$.

**Remark 5.4.** Taking the expectation on both sides of (5.25) and using property 4 in Theorem 5.3 we obtain

$$\mathbb{E}[X_{i+1}] = \mathbb{E}[X_i], \quad \text{for all } i \geq 1.$$

Thus, iterating, $\mathbb{E}[X_i] = \mathbb{E}[X_1]$, for all $i \geq 1$, i.e., *martingales have constant expectation.*

An example of martingale is the random walk introduced above. In fact, using the linearity of the conditional expectation we have

$$\begin{aligned}\mathbb{E}[M_n|M_1,\dots,M_{n-1}] &= \mathbb{E}[M_{n-1}+X_n|M_1,\dots,M_{n-1}]\\ &= \mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}] + \mathbb{E}[X_n|M_1,\dots,M_{n-1}].\end{aligned}$$

As $M_{n-1}$ is measurable with respect to $M_1,\dots,M_{n-1}$, then $\mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}] = M_{n-1}$, see Theorem 5.3 (3). Moreover, as $X_n$ is independent of $M_1,\dots,M_{n-1}$, Theorem 5.3 (2) gives $\mathbb{E}[X_n|M_1,\dots,M_{n-1}] = \mathbb{E}[X_n] = 0$. It follows that $\mathbb{E}[M_n|M_1,\dots,M_{n-1}] = M_{n-1}$, i.e., the random walk is a martingale.

## 5.4 Applications to the binomial model

In this section we reformulate the binomial options pricing model using the language of probability theory. We first show that the binomial stock price can be interpreted as a stochastic process defined on the $N$-coin toss probability space $(\Omega_N, \mathbb{P}_p)$, see Definition 5.2. Recall that, for a given $0 < p < 1$, the binomial stock price at time $t$ is given by

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p\\ S(t-1)e^d & \text{with probability } 1-p,\end{cases} \tag{5.26}$$

where $u > d$. Now, for $t \in \mathcal{I} = \{1,\dots,N\}$, consider the following random variable

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \begin{cases} u, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H\\ d, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T\end{cases}.$$

Hence we can write

$$S(t) = S(t-1)e^{X_t} = S(t-2)e^{X_{t-1}+X_t} = \cdots = S_0\exp(X_1+X_2+\cdots+X_t) : \Omega_N \to \mathbb{R},$$

where $S(0)$ is the initial value of the stock price, which is a deterministic constant. Thus $S(t)$ is a random variable and therefore $\{S(t)\}_{t\in\mathcal{I}}$ is a (discrete) stochastic process. For each $\omega \in \Omega_N$, the vector $(S(1,\omega),\dots,S(N,\omega))$ is a path for the stock price.

The value at time $t$ of the risk-free asset is the deterministic function of time $B(t) = B_0\exp(rt)$, where $r$ is the (constant) interest rate of the money market and $B_0$ is the initial value of the risk-free asset. Recall that $S^*(t) = e^{-rt}S(t)$ is called the discounted price of the stock, see Remark 2.3.

**Theorem 5.4.** *If $r \notin (d,u)$, there is no probability measure $\mathbb{P}_p$ on the sample space $\Omega_N$ such that the discounted stock price process $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale. For $r \in (d,u)$, $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where*

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

*Proof.* By definition, $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale if and only if

$$\mathbb{E}[e^{-rt}S(t)|S^*(1),\dots,S^*(t-1)] = e^{-r(t-1)}S(t-1), \quad \text{for all } t\in\mathcal{I}.$$

Clearly, taking the expectation conditional to $S^*(1),\dots,S^*(t-1)$ is the same as taking the expectation conditional to $S(1),\dots,S(t-1)$, hence the above equation is equivalent to

$$\mathbb{E}[S(t)|S(1),\dots,S(t-1)] = e^{r}S(t-1), \quad \text{for all } t\in\mathcal{I}, \tag{5.27}$$

where we canceled out a factor $e^{-rt}$ in both sides of the equation. Moreover

$$\begin{aligned}\mathbb{E}[S(t)|S(1),\dots,S(t-1)] &= \mathbb{E}[\frac{S(t)}{S(t-1)}S(t-1)|S(1),\dots,S(t-1)]\\ &= S(t-1)\mathbb{E}[\frac{S(t)}{S(t-1)}|S(1),\dots,S(t-1)],\end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditioning variables and thus can be taken out from the conditional expectation (see property 5 in Theorem 5.3). As

$$S(t)/S(t-1) = \begin{cases} e^u & \text{with prob. } p\\ e^d & \text{with prob. } 1-p\end{cases}$$

is independent of $S(1),\dots,S(t-1)$, then by Theorem 5.3(2) we have

$$\mathbb{E}[\frac{S(t)}{S(t-1)}|S(1),\dots,S(t-1)] = \mathbb{E}[\frac{S(t)}{S(t-1)}] = e^u p + e^d(1-p).$$

Hence (5.27) holds if and only if $e^u p + e^d(1-p) = e^r$. The latter has a solution $p\in(0,1)$ if and only if $r\in(d,u)$ and the solution, when it exists, is unique and given by $p=q$. □

**Remark 5.5.** Due to Theorem 5.4, $\mathbb{P}_q$ is called **martingale probability measure**. Moreover we can reformulate Theorem 2.3 as follows: *a 1+1 dimensional binomial stock market is arbitrage free if and only if there exists a martingale probability measure.*

Since martingales have constant expectation (see Remark 5.4), we obtain the important result

$$\mathbb{E}_q[S(t)] = S_0 e^{rt}. \tag{5.28}$$

Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk neutral probability**. However, as shown in the following exercise, the situation is very different in the physical (real world) probability.

**Exercise 5.19 (•).** *Let* $\mathbb{E}_p[\cdot]$ *denote the expectation in the probability measure* $\mathbb{P}_p$. *Show that*

$$\mathbb{E}_p[S(N)] = S(0)(e^u p + e^d(1-p))^N, \quad \mathbb{E}_q[S(N)] = S(0)e^{rN}.$$

*Compute also* $\mathrm{Var}_p[S(N)]$.

The value of a portfolio position $(h_S, h_B)$ invested on $h_S$ shares of the stock and $h_B$ shares of the risk-free asset is the stochastic process $\{V(t)\}_{t\in\mathcal{I}}$, where

$$V(t) = h_B B(t) + h_S S(t) \, : \, \Omega_N \to \mathbb{R}, \quad t \in \mathcal{I},$$

while $V(0) = h_S S_0 + h_B B_0$ is a deterministic constant. If we change the portfolio position depending on the price of the stock (i.e., depending on $\omega \in \Omega_N$), then we get a portfolio (stochastic) process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$. We recall that $(h_S(t), h_B(t))$ corresponds to the portfolio position held in the interval $(t-1, t]$. The portfolio process is predictable if $h_S(t), h_B(t)$ are measurable with respect to $S(1), \dots, S(t-1)$. The value $\{V(t)\}_{t\in\mathcal{I}}$ of the portfolio process is the stochastic process given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t) \, : \, \Omega_N \to \mathbb{R}, \quad t \in \mathcal{I}. \tag{5.29}$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ is said to be self-financing if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1). \tag{5.30}$$

**Theorem 5.5** (∗). *Let $\{(h_S(t), h_B(t))\}_{t\in\mathcal{I}}$ be a self-financing predictable portfolio with value $\{V(t)\}_{t\in\mathcal{I}}$. Then the discounted portfolio value $V^*(t) = e^{-rt}V(t)$ is a martingale in the risk-neutral measure. Moreover the following identity holds:*

$$V^*(t) = \mathbb{E}_q[V^*(N)|S(1), \dots, S(t)]. \tag{5.31}$$

*In particular at time $t = 0$ we obtain $V(0) = e^{-rN}\mathbb{E}_q[V(N)]$, which is equivalent to* (2.10).

*Proof.* The martingale claim is

$$\mathbb{E}_q[V^*(t)|V^*(1), \dots, V^*(t-1)] = V^*(t-1).$$

We now show that this follows by

$$\mathbb{E}_q[V^*(t)|S(1), \dots, S(t-1)] = V^*(t-1). \tag{5.32}$$

In fact, by taking the conditional expectation in the risk-neutral measure of (5.32) with respect to $V^*(1), \dots, V^*(t-1)$, we obtain

$$\begin{aligned} V^*(t-1) &= \mathbb{E}_q[V^*(t-1)|V^*(1), \dots, V^*(t-1)] \\ &= \mathbb{E}_q\big[\mathbb{E}_q[V^*(t)|S(1), \dots, S(t-1)]|V^*(1), \dots, V^*(t-1)\big] \\ &= \mathbb{E}_q[V^*(t)|V^*(1), \dots, V^*(t-1)]; \end{aligned}$$

here we have used property 3 of Theorem 5.3 in the first step and property 6 in the last step. The latter is possible because $V^*(t)$ is measurable with respect to $S(1), \dots, S(t)$ (being the portfolio process predictable). Now we claim that (5.32) also implies the formula (5.31). We argue by backward induction[3]. Letting $t = N$ in (5.32) we see that (5.31) holds at $t = N-1$. Assume now that holds (5.31) at time $t+1$, i.e.,

$$V^*(t+1) = \mathbb{E}_q[V^*(N)|S(1), \dots, S(t+1)].$$

Taking the conditional expectation with respect to $S(1), \dots, S(t)$ in the risk neutral measure we have, by (5.32),

$$\begin{aligned} V^*(t) &= \mathbb{E}_q[V^*(t+1)|S(1), \dots, S(t)] = \mathbb{E}_q\Big[\mathbb{E}_q[V^*(N)|S(1), \dots, S(t+1)]|S(1), \dots, S(t)\Big] \\ &= \mathbb{E}_q[V^*(N)|S(1), \dots, S(t)], \end{aligned}$$

hence (5.32) $\Rightarrow$ (5.31), as claimed. It remains to prove (5.32). As $B(t) = B(t-1)e^r$, (5.30) gives

$$h_B(t)B(t) = e^r V(t-1) - h_S(t)S(t-1)e^r.$$

Replacing in (5.29) we find

$$V(t) = e^r V(t-1) + h_S(t)[S(t) - S(t-1)e^r].$$

Taking the conditional expectation in the risk neutral measure with respect to the random variables $S(1), \dots, S(t-1)$ we obtain

$$\begin{aligned} \mathbb{E}_q[V(t)|S(1), \dots S(t-1)] &= e^r \mathbb{E}_q[V(t-1)|S(1), \dots, S(t-1)] \\ &\quad + \mathbb{E}_q[h_S(t)(S(t) - S(t-1)e^r)|S(1), \dots, S(t-1)]. \end{aligned} \tag{5.33}$$

As $V(t-1)$ and $h_S(t)$ are measurable with respect to the conditioning variables we have $\mathbb{E}_q[V(t-1)|S(1), \dots, S(t-1)] = V(t-1)$, as well as

$$\begin{aligned} &\mathbb{E}_q[h_S(t)(S(t) - S(t-1)e^r)|S(1), \dots, S(t-1)] \\ &= h_S(t)\mathbb{E}_q[S(t) - S(t-1)e^r|S(1), \dots, S(t-1)] \\ &= h_S(t)\Big(\mathbb{E}_q[S(t)|S(1), \dots, S(t-1)] - S(t-1)e^r\Big) = 0, \end{aligned}$$

where in the last step we used that $\{S^*(t)\}_{t\in\mathcal{I}}$ is a martingale in the risk-neutral measure. Going back to (5.33) we obtain

$$\mathbb{E}_q[V(t)|S(1), \dots S(t-1)] = e^r V(t-1),$$

which is the same as (5.32). □

Now we can use the martingale property of $\{V^*(t)\}_{t\in\mathcal{I}}$ to give a simple proof that the existence of a martingale probability implies the absence of arbitrage. In fact, assume that $\{h_S(t), h_B(t)\}_{t\in\mathcal{I}}$ is an arbitrage (see Definition 2.4). Then $V^*(0) = 0$ and since martingales have constant expectation then $\mathbb{E}[V^*(t)] = 0$, for all $t \in \{0, 1 \dots, N\}$. But $V^*(N) \geq 0$, hence

[3] Note the similarity of this proof with the one of Theorem 2.2.

by 2 of Theorem 5.1 it must be $V^*(N,x) = 0$ along any path $x \in \{u,d\}^N$. Thus the portfolio is not an arbitrage.

Now let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European-style derivative with pay-off $Y : \Omega_N \to \mathbb{R}$ at maturity time $N$. This means that the derivative can only be exercised at time $t = N$ (for standard European derivatives $Y$ is a deterministic function of $S(N)$). Let $\Pi_Y(t)$ be the binomial fair price of the derivative a time $t$. By definition, $\Pi_Y(t)$ equals the value $V(t)$ of self-financing, hedging portfolios. In particular, $\Pi_Y(t)$ is a random variable and so $\{\Pi_Y(t)\}_{t\in\mathcal{I}}$ is a stochastic process. Using the hedging condition $V(N) = Y$ (which means $V(N,\omega) = Y(\omega)$, for all $\omega \in \Omega_N$)) and (5.31), we have the following formula for the fair price at time $t$ of the financial derivative:

$$\Pi_Y(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(1),\dots,S(t)]. \tag{5.34}$$

Equation (5.34) is known as **risk neutral pricing formula** and it is the cornerstone of options pricing theory. It holds not only for the binomial model but for any discrete—or even continuum—pricing model for financial derivatives. It is used for standard as well as non-standard European derivatives. In the special case $t = 0$, (5.34) reduces to

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y]. \tag{5.35}$$

**Exercise 5.20 (Put-call parity for Asian options (•)).** *Consider a $N$-period arbitrage-free binomial market with $r \neq 0$ and let $S(t)$ denote the price of the stock at time $t \in \{0,\dots,N\}$. The Asian call, resp. put, with maturity $T = N$ and strike $K$ is the non-standard European style derivative with pay-off*

$$Y_{\text{call}} = \left[\Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big) - K\right]_+, \quad \textit{resp.} \quad Y_{\text{put}} = \left[K - \Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big)\right]_+ .$$

*Denote by $AC(0)$ and $AP(0)$ the binomial price at time $t = 0$ of the Asian call and put, respectively. Prove the following put-call parity identity:*

$$AC(0) - AP(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\frac{e^{r(N+1)}-1}{e^r-1} - K\Big].$$

*HINT: For $\alpha \neq 1$, $\sum_{k=0}^{N}\alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$.*

### 5.4.1 Binomial distribution

Let us now derive the distribution of the stock binomial price $S(t)$, $t \in \mathcal{I}$. First we notice that the image of $S(t)$ is $\operatorname{Im} S(t) = \{s_0,\dots,s_t\}$, where

$$s_k = S(0)\exp(ku + (t-k)d), \quad k = 0,\dots,t.$$

Hence $f_{S(t)}(x) = 0$, if $x \neq s_k$, for all $k = 0,\dots t$ and $t \in \mathcal{I}$. The value of $S(t)$ is $s_k$ if there are exactly $k$ heads in the first $t$ tosses of the $N$-toss, which shows that $f_{S(t)}$ is given by the so-called **binomial distribution**:

$$f_{S(t)}(s_k) = \mathbb{P}_p[S(t) = s_k] = \binom{t}{k} p^k (1-p)^{t-k}, \quad k = 0, \dots, t, \quad t \in \mathcal{I}.$$

In particular, in the case of a fair coin $p = 1/2$, we have

$$f_{S(t)}(x) = \binom{t}{k} \frac{1}{2^t} \delta(x - S(0)\exp(ku + (t-k)d)) \quad \text{(fair coin)},$$

where $\delta(z) = 1$ if $z = 0$ and $\delta(z) = 0$ for $z \neq 0$. It is convenient to express this result in terms of the log-price of the stock $\log S(t)$, as this can take both positive and negative values. For a fair coin we get

$$f_{\log(S(t)/S(0))}(x) = \binom{t}{k} \frac{1}{2^t} \delta(x - (ku + (t-k)d). \tag{5.36}$$

An example of this distribution is depicted in Figure 5.2. As it is clear from the picture, for large values of $N \in \mathbb{N}$, the distribution of $\log(S(N)/S(0)) = \log S(N) - \log S(0)$ can be approximated by a **normal** distribution, i.e., a distribution of the form (5.39) below[4]. One of the main critics to the binomial model is that it assigns very low probabilities to large variations of the (log-)stock price.

The binomial distribution can be used to compute the probability that the price of the stock lies in an interval $[a, b]$ at time $t$. By (5.7) we have

$$\mathbb{P}(a \leq S(t) \leq b) = \sum_{k: a \leq s_k \leq b} f_{S(T)}(s_k) = \sum_{k: a \leq s_k \leq b} \binom{t}{k} p^k (1-p)^{t-k} \quad k = 0, \dots, t, \quad t \in \mathcal{I}. \tag{5.37}$$

Moreover if $g : \mathbb{R} \to \mathbb{R}$ then, by (5.8),

$$\mathbb{P}(a \leq g(S(t)) \leq b) = \sum_{k: a \leq g(s_k) \leq b} f_{S(T)}(s_k) = \sum_{k: a \leq g(s_k) \leq b} \binom{t}{k} p^k (1-p)^{t-k} \quad k = 0, \dots, t, \quad t \in \mathcal{I}. \tag{5.38}$$

The formula (5.38) can be used to compute the probability that a derivative on the stock is in the money at time $t$. Consider a standard European or American derivative with pay-off $Y = g(S(N))$ at the expiration date $T = N$. Then the probability that the derivative is in the money at time $t$ is given by

$$\mathbb{P}(Y(t) > 0) = \mathbb{P}(g(S(t)) > 0) = \sum_{k: g(s_k) > 0} \binom{t}{k} p^k (1-p)^{t-k}.$$

These formulas can be easily implemented with Matlab.

**Exercise 5.21 (Matlab).** *Write a Matlab function ProbDerivative(S,t,g,p) which computes the probability that a standard derivative with pay-off function g is in the money at time t.*

[4]A rigorous proof of the fact that the distribution of $\log S(N)/S(0)$ converges to a normal distribution in the time-continuum limit is given in Theorem 6.3.

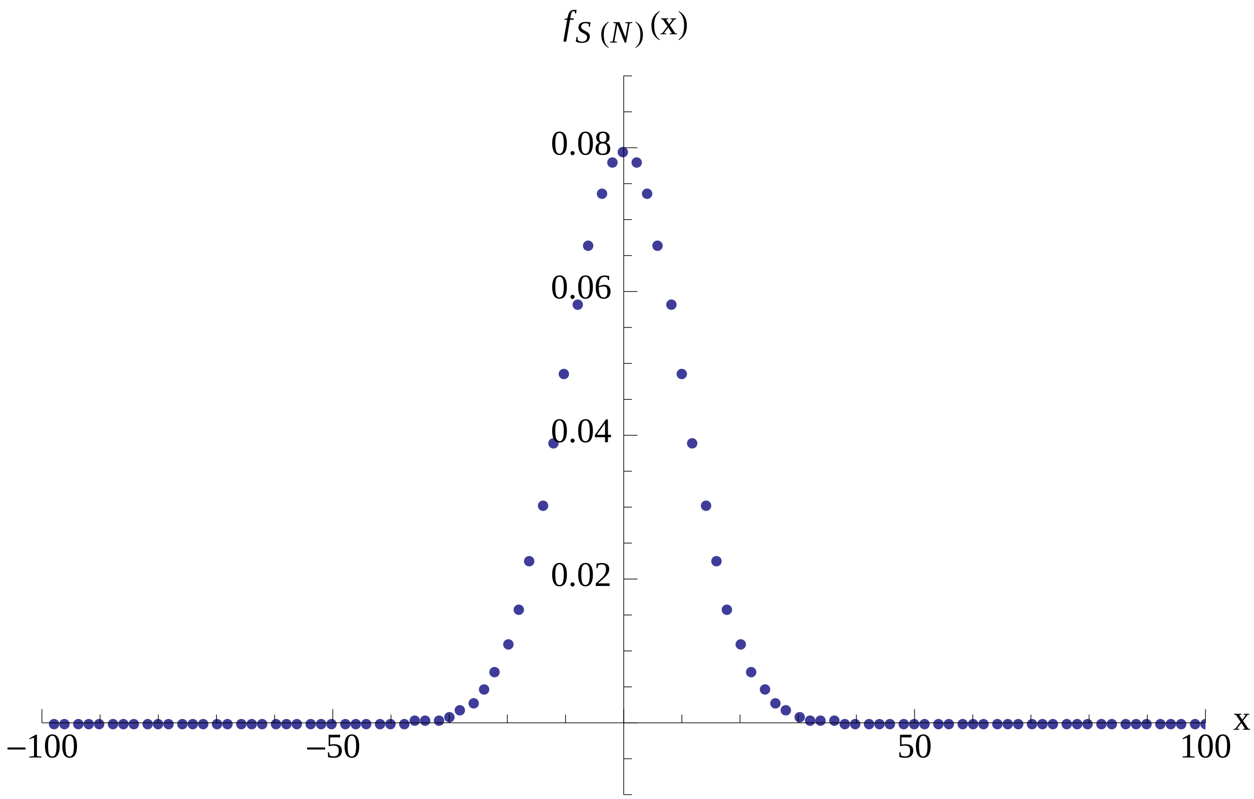



Figure 5.2: The distribution of $\log(S(N))$. It is assumed $S(0) = 1$, $u = -d = 1$, $p = 1/2$ and $N = 100$.

### 5.4.2 General discrete options pricing models

Equation (5.34) is the most important result of this chapter. It leads to the following general strategy to introduce discrete options pricing models for European derivatives:

1. Fix a finite probability space $(\Omega, \mathbb{P}(\lambda))$. The probability measures depends on a set of parameters $\lambda = (\lambda_1, \dots, \lambda_n)$ (e.g., $\lambda = p$ in the $N$-coin toss probability space)

2. Define the stock price as a positive stochastic process on the probability space $(\Omega, \mathbb{P}(\lambda))$.

3. Prove the existence of a probability measure $\mathbb{Q} = \mathbb{P}(\lambda_0)$ which makes the stock price process a martingale. The existence of a martingale measure ensures that the market is arbitrage-free

4. Provided the martingale measure is unique (i.e., provided $\lambda_0$ is unique), define the fair price of European derivatives by (5.34). If the martingale measure is not unique, then the price of the derivative is not uniquely defined and the model is called **incomplete**.

In fact, the above strategy is applied for *all* models in options pricing theory, even for time-continuum models, which are defined on uncountable probability spaces. An example of time-continuum model is the Black-Scholes model considered in the next chapter (see [7] for an introduction to general time-continuum models). The important lesson to be learned here is the following: *probability theory is the right framework where to formulate the mathematical models in options pricing theory.*

**Exercise 5.22.** *Formulate a discrete options pricing model in which, at any time step, the stock price can go up, down or stay the same (***trinomial model***). Use the language of probability theory. Show that this model is incomplete (i.e., the martingale measure is not unique). HINT: For the last part of the exercise one can restrict to the one-period model.*

### 5.4.3 Quantitative vs fundamental analysis of a stock

To continue this section we discuss briefly the difference between the **fundamental** analysis and the **quantitative** (or statistical) analysis of a stock. Performing a quantitative analysis means that the investor tries to estimate the price of the stock using a mathematical probabilistic model, such as the binomial model. As opposed to this, a fundamental analysis of the stock consists in a careful evaluation of the performance of the underlying company and of other economical factors which may affect the value of the stock. As the quantitative analysis, also the fundamental analysis can be formulated in the language of probability theory (although in practice nobody does that!). Precisely, let $S(T)$ be the price of the stock at some future time $T > 0$ (e.g., $T = 1$ year from now) and let us start by making a list of all events which we think may affect the value of $S(T)$. Let us call $\{\omega_1\}, \{\omega_2\}, \dots, \{\omega_M\}$ such events; for instance

$$\begin{aligned}
&\{\omega_1\} = [\text{the company will make a profit this year}] \\
&\{\omega_2\} = [\text{there will be an oil crisis this year}] \\
&\{\omega_3\} = [\text{the company will open a new branch}] \\
&\text{etc.}
\end{aligned}$$

In this way we have constructed a sample space $\Omega = \{\omega_1, \dots, \omega_M\}$. Next we have to assign a probability to each of these atomic events, say $\mathbb{P}(\{\omega_i\}) = p_i$. Finally we have to specify how these events affect the price of the stock, that is to say, we have to give $S(T, \omega)$, for $\omega \in \Omega$. After all of this we can compute for instance

$$\mathbb{E}[S(T)] = \sum_{\omega \in \Omega} S(T, \omega)\mathbb{P}(\{\omega\}), \quad \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2,$$

which are our "expected value" for the stock price and our "expected error" of this value. Clearly all the steps that led us to these estimates are quite subjective.

In the quantitative analysis we only try to "guess" what the distribution of the stock price will be at time $T$, i.e., we assign $f_{S(T)} : (0, \infty) \to [0, 1]$ and then we derive our estimates using the formulas

$$\mathbb{E}[S(T)] = \sum_{s \in I} s\, f_{S(T)}(s), \quad \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2,$$

where $I$ is a (finite) set of possible values that we admit for the stock price. Note that in the quantitative approach the only arbitrariness is in the choice of the distribution function of the price. In the binomial model we assume that $\log S(T)$ follows a binomial distribution, but other choices are possible and can be justified by looking at the historical data for the stock price. In fact, the advantage of the quantitative method versus the fundamental approach, is that, by employing the stock price as the only relevant information, it provides us with an objective (quantitative) way to justify, monitor and adjust our analysis.

## 5.5 Infinite Probability Spaces

In this last section we consider briefly probability spaces consisting of an infinite number of sample points. The discussion can be complemented with Chapter 4 in [3].

Assume first that $\Omega$ is a countable set. This means that

$$\Omega = \{\omega_n\}_{n\in\mathbb{N}}.$$

For countable sample spaces the definitions given in the previous sections for finite sets extend straightforwardly. Precisely, given a sequence

$$p = (p_n)_{n\in\mathbb{N}} \quad \text{such that} \quad 0 < p_n < 1, \quad \sum_{n\in\mathbb{N}} p_n = 1,$$

we define the probability of the atomic events as

$$\mathbb{P}(\{\omega_n\}) = p_n.$$

If $A \in 2^\Omega$, then we define

$$\mathbb{P}(A) = \sum_{i:\omega_i\in A} p_i = \sum_{\omega\in A} \mathbb{P}(\{\omega\}).$$

If $X : \Omega \to \mathbb{R}$ is a random variable, then

$$\mathbb{E}[X] = \sum_{n\in\mathbb{N}} X(\omega_n)p_n = \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\{\omega\}).$$

The remaining definitions introduced in the finite case (variance, covariance, independent random variables, etc.) continue to be valid for countable probability spaces.

When $\Omega$ is uncountable, there is no general procedure to construct a probability space, but only an abstract definition (which we shall not give). The problem is that in the uncountable case $\Omega$ admits very irregular ("wild") sets and thus defining a probability over the whole $2^\Omega$ becomes complicated. Moreover the occurrence of non-trivial events with zero probability poses some technical problems. We restrict ourselves to present some examples.

- Let $\Omega = (0, 1)$ and let the admissible events be given by the sets which can be written as the union of countably many open subintervals of $(0, 1)$. For any admissible event $A \subseteq (0, 1)$ we define
$$\mathbb{P}(A) = \int_0^1 \mathbb{I}_A(x)\, dx,$$
where $\mathbb{I}_A(x) = 1$ if $x \in A$ and zero otherwise.

- Let $\Omega = \mathbb{R}$ and $p : \mathbb{R} \to [0, \infty)$ be a continuous function such that
$$\int_{\mathbb{R}} p(x)\, dx = 1.$$
As admissible events we consider all real sets which can be written as the union of countably many open intervals. For any admissible event $A \in \mathbb{R}$ we define
$$\mathbb{P}(A) = \int_A p(x)\, dx.$$

- In this example we construct a probability space that extends $(\Omega_N, \mathbb{P}_p)$ (the $N$-coin toss probability space) to $N = \infty$. Consider $\Omega_\infty = \{(\gamma_n)_{n\in\mathbb{N}};\, \gamma_i = H \text{ or } T\}$. Thus an element of $\Omega_\infty$ is the outcome of the experiment "tossing a coin infinitely many times". It can be shown (using a standard Cantor diagonal argument), that the set $\Omega_\infty$ is uncountable. Now, given $\bar{\omega} = (\bar{\gamma}_1, \dots, \bar{\gamma}_N) \in \Omega_N$, consider the events
$$A_N(\bar{\omega}) \subset \Omega_\infty, \quad A = \{(\gamma_n)_{n\in\mathbb{N}} \in \Omega_\infty : \gamma_i = \bar{\gamma}_i,\, i = 1, \dots, N\},$$
that is to say, $A_N(\bar{\omega})$ contains all infinity-tosses whose first $N$-tosses coincide with $(\bar{\gamma}_1, \dots, \bar{\gamma}_N)$. Of course, $A_N(\bar{\omega})$ is also uncountable. We define the probability of $A_N(\bar{\omega})$ as the probability of $(\bar{\gamma}_1, \dots, \bar{\gamma}_N)$ in the probability space $(\Omega_N, \mathbb{P}_p)$. For instance, assuming that the coin is fair, the event
$$\text{[the first 5 tosses in the infinite sequence are heads]}$$
has probability $(1/2)^5$. By letting $\bar{\omega}$ varies in $\Omega_N$, we get a collection $\mathcal{A}_N$ of $2^N$ events in $\Omega_\infty$,
$$\mathcal{A}_N \subset 2^{\Omega_\infty}, \quad \mathcal{A}_N = \{A_N(\bar{\omega});\, \bar{\omega} \in \Omega_N\},$$
whose probability has been defined. This definition can be extended to all events that can be written as the disjoint union of events in the set $\mathcal{A}_N$, using the rule $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if $A, B$ are disjoint. In this way we have assigned a probability to all events in the space $\Omega_\infty$ that depends only on the first $N$ tosses, and letting $N$ arbitrarily large, we are able to define a probability to any event which depends on a (arbitrary) finite number of tosses (e.g., the event "there is a tail in the first $10^{100}$ tosses"). This probability space might seem quite large, but actually is not! For instance, the event "there exists at least one head in the infinite-toss" does not belong

to this space, hence we cannot assign a probability to it. The inclusion of events which are resolved by tossing the coin infinitely many times requires advanced tools in probability theory (in particular, Carathéodory's theorem), which will not be discussed here.

Fortunately for most applications (and in particular for those in financial mathematics) the knowledge of the full probability space is usually not necessary, as in the applications one is typically concerned only with random variables and their distributions, rather than with generic events. More precisely, we are only interested in assigning a probability to events of the form $\{X \in I\}$, where $X$ is a random variable on the (abstract) probability space and $I \subset \mathbb{R}$, that is to say, events which can be resolved by one (or more) random variables, cf. the discussion in Section 5.4.3.

**Definition 5.16.** *Let $f_X : \mathbb{R} \to [0, \infty)$ be a continuous function, except possibly on finitely many points. A random variable $X : \Omega \to \mathbb{R}$ is said to have* **probability density** *$f_X$ if*

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx,$$

*for all regular sets $A \subseteq \mathbb{R}$.*

Note that if $f_X$ is the probability density of a random variable $X$, then necessarily

$$\int_{\mathbb{R}} f_X(x)\, dx = 1.$$

Moreover the cumulative distribution $F_X(x) = \mathbb{P}(X \le x)$ satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy, \quad \text{for all } x \in \mathbb{R}, \text{ hence } f_X = \frac{dF_X}{dx}.$$

### Examples

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **normal**, or to have a normal distribution, with **mean** $m \in \mathbb{R}$ and **variance** $\sigma^2 > 0$ if

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-m|^2}{2\sigma^2}\right). \tag{5.39}$$

  We denote $\mathcal{N}(m, \sigma^2)$ the set of all such random variables. A variable $X \in \mathcal{N}(0,1)$ is called a **standard** normal random variable. The cumulative distribution of standard normal random variables is denoted by $\Phi(x)$ and is called the **standard normal distribution**, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

- A random variable $X : \Omega \to \mathbb{R}$ is said to be **exponential**, or to have an exponential distribution, with **intensity** $\lambda > 0$ if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

We denote $\mathcal{E}(\lambda)$ the set of all exponential random variables with intensity $\lambda$.

### 5.5.1 Joint distribution. Independence

**Definition 5.17.** *Two random variables $X, Y : \Omega \to \mathbb{R}$ are said to have the* **joint probability density** $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$, *if*

$$\mathbb{P}(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x, y)\, dx\, dy,$$

*for all regular sets $A, B \subseteq \mathbb{R}$.*

The generalization of the previous definition to $n$ variables is straightforward. Note that if $f_{X,Y}$ is a joint probability density, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_{X,Y}(x, y)\, dx\, dy = 1.$$

**Example: Jointly normally distributed random variables.** Let $m \in \mathbb{R}^2$ and $C = (C_{ij})_{i,j=1,2}$ be a symmetric, positive definite 2×2 matrix. Two random variables $X_1, X_2 : \Omega \to \mathbb{R}$ are said to be jointly normally distributed with mean $m$ and **covariance matrix** $C$ if they admit the joint density

$$f_{X_1,X_2}(x) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left( -\frac{1}{2}(x - m) \cdot C^{-1} \cdot (x - m)^T \right), \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2, \tag{5.40}$$

where " · " denotes the row-by-column product of matrices. This definition extends straightforwardly to $n$ variables.

The following theorem, which we give without proof, shows that the probability density, when it exists, provides all the relevant statistical information on a random variable.

**Theorem 5.6.** *The following holds for all sufficiently regular*[5] *functions $g : \mathbb{R} \to \mathbb{R}$:*

*(i) Let $X : \Omega \to \mathbb{R}$ be a random variable with density $f_X$. Then for all regular sets $A \subseteq \mathbb{R}$,*

$$\mathbb{P}(g(X) \in A) = \int_{x: g(x) \in A} f_X(x)\, dx,$$

*which extends* (5.8) *to general probability spaces.*

[5] In particular, for all functions $g$ such that the integrals in the theorem are well-defined.

*(ii) Let $X : \Omega \to \mathbb{R}$ be a random variable with density $f_X$. Then*

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(y) f_X(y)\, dy,$$

*which extends* (5.13) *to general probability spaces.*

*(iii) Let $X, Y : \Omega \to \mathbb{R}$ be random variables with joint density $f_{X,Y}$. Then*

$$\mathbb{E}[XY] = \int_{\mathbb{R}^2} xy f_{X,Y}(x, y)\, dx\, dy,$$

*which extends* (5.20) *to general probability spaces.*

As an example of application of (i), suppose $X \in \mathcal{N}(0, 1)$. Then

$$\mathbb{P}(X^2 \leq 1) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{x^2}{2}}\, dx \approx 0.683,$$

that is to say, a standard normal random variable has about 68,3% probability to take value on the interval $(-1, 1)$. Let us see some further applications of Theorem 5.6. By (ii), the expectation and the variance of a random variable $X$ with density $f_X$ are given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\, dx, \quad \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_{\mathbb{R}} x^2 f_X(x)\, dx - \left( \int_{\mathbb{R}} x f_X(x)\, dx \right)^2.$$

These formulas generalize (5.11) and (5.15) to random variables on general sample spaces which admit a density distribution. In particular, for normal variables we obtain

$$X \in \mathcal{N}(m, \sigma^2) \implies \mathbb{E}[X] = m, \quad \mathrm{Var}[X] = \sigma^2. \tag{5.41}$$

**Exercise 5.23.** *Prove* (5.41). *Compute the expectation and the variance of exponential random variables.*

By (iii) of Theorem 5.6, if $X_1, X_2$ have the joint density $f_{X_1,X_2}$, then

$$\mathrm{Cov}(X_1, X_2) = \int_{\mathbb{R}^2} x_1 x_2 f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 - \mathbb{E}[X_1]\mathbb{E}[X_2],$$

which generalizes (5.21). In particular, if $X_1, X_2$ are jointly normal distributed with mean $m \in \mathbb{R}^2$ and covariance matrix $C = (C_{ij})_{i,j=1,2}$, we find

$$m = (m_1, m_2), \qquad C_{ij} = \mathrm{Cov}(X_i, X_j). \tag{5.42}$$

**Exercise 5.24.** *Prove* (5.42).

The next thing we need to know about general probability spaces is how to determine when two random variables are independent. To this regard we have the following theorem.

**Theorem 5.7** (∗). *The following holds.*

*(i) If two random variables $X, Y$ admit densities $f_X$, $f_Y$ and are independent, then they admit the joint density*

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

*(ii) If two random variables $X, Y$ admit a joint density $f_{X,Y}$ of the form*

$$f_X(x,y) = u(x)v(y),$$

*for some functions $u, v : \mathbb{R} \to [0,\infty)$, then $X, Y$ are independent and admit densities $f_X$, $f_Y$ given by*

$$f_X(x) = cu(x), \quad f_Y(y) = \frac{1}{c}v(y),$$

*where*

$$c = \int_{\mathbb{R}} v(x)\,dx = \Big(\int_{\mathbb{R}} u(y)\,dy\Big)^{-1}.$$

*Proof.* As to (i) we have

$$\begin{aligned}\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) &= \int_A f_X(x)\,dx \int_B f_Y(y)\,dy \\ &= \int_{A\times B} f_X(x)f_Y(y)\,dx\,dy.\end{aligned}$$

To prove (ii), we first write

$$\{X \in A\} = \{X \in A\} \cap \Omega = \{X \in A\} \cap \{Y \in \mathbb{R}\} = \{X \in A, Y \in \mathbb{R}\}.$$

Hence, by definition of joint density,

$$\mathbb{P}(X \in A) = \int_A \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_A u(x)\,dx \int_{\mathbb{R}} v(y)\,dy = \int_A cu(x)\,dx$$

where $c = \int_{\mathbb{R}} v(x)\,dx$. Thus $X$ admits the density $f_X(x) = c\,u(x)$. At the same fashion one proves that $Y$ admits the density $f_Y(y) = c'v(y)$, where $c' = \int_{\mathbb{R}} u(y)dy$. Since

$$1 = \int_{\mathbb{R}}\int_{\mathbb{R}} f_{X,Y}(x,y)\,dx\,dy = \int_{\mathbb{R}} u(x)\,dx \int_{\mathbb{R}} v(y)\,dy = c'c,$$

then $c' = 1/c$. It remains to prove that $X, Y$ are independent. This follows by

$$\begin{aligned}\mathbb{P}(X \in A, Y \in B) &= \int_A\int_B f_{X,Y}(x,y)\,dx\,dy = \int_A u(x)\,dx \int_B v(y)\,dy \\ &= \int_A cu(x)\,dx \int_B \frac{1}{c}v(y)\,dy = \int_A f_X(x)\,dx \int_B f_Y(y)\,dy \\ &= \mathbb{P}(X \in A)\mathbb{P}(Y \in B).\end{aligned}$$

□

**Exercise 5.25.** *Let $X \in \mathcal{N}(0,1)$ and $Y \in \mathcal{E}(1)$ be independent. Compute $\mathbb{P}(X \leq Y)$.*

As an application of Theorem 5.7 we now prove the following result.

**Theorem 5.8.** *Let $X_1, X_2 : \Omega \to \mathbb{R}$ be normal random variables which are jointly normally distributed. Then the following properties are equivalent:*

*(a) $X_1, X_2$ are independent;*

*(b) $X_1, X_2$ are uncorrelated.*

*Proof.* (a)$\Longrightarrow$(b) is always true, for all random variables. As to the implication (b)$\Longrightarrow$(a), by (5.42) we have $C_{12} = C_{21} = 0$. Substituting in (5.40) we obtain the $f_{X_1,X_2}$ has the form $f_{X1,X_2}(x_1, x_2) = u(x_1)v(x_2)$, and so the claim follows by (ii) of Theorem 5.7. □

The next theorem shows that independent normal random variables form a linear space.

**Theorem 5.9.** *If $X_1, X_2$ are independent random variables such that $X_i \in \mathcal{N}(\alpha_i, \sigma_i^2)$, $i = 1, 2$, then $X_1 + X_2 \in \mathcal{N}(\alpha_1 + \alpha_2, \sigma_1^2 + \sigma_2^2)$.*

A way to prove Theorem 5.9 is suggested in the additional exercises for Chapter 5 in Appendix D.

### 5.5.2 Central limit theorem

The normal distribution is certainly the most important random variables distribution used in finance (and in many other applications of probability theory). This is partially due to historical reasons, but also to the following fundamental result.

**Theorem 5.10 (Central Limit Theorem).** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}[X_i] = 1$. Let $S_n$ be the* **sample average** *of the first $n$ random variables, i.e.,*

$$S_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

*Then, as $n \to \infty$, $\sqrt{n}S_n$ converges in distribution to a standard normal random variable, that is*

$$\lim_{n\to\infty} F_{\sqrt{n}S_n}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy, \quad \text{for all } x \in \mathbb{R}.$$

For example, Theorem 5.10 applied to the random walk $\{M_n\}_{n\in\mathbb{N}}$ implies that in the limit $n \to +\infty$, the random variable $M_n/\sqrt{n}$ converges in distribution to a standard normal random variable, which is consistent with the result of Exercise 5.17 and the subsequent remark that the stochastic process $\{M_{[nt]}/\sqrt{n}\}_{t\in[0,T]}$ approximates a Brownian motion in the limit $n \to \infty$ (see Definition 5.18 below). The proof of the Central Limit Theorem can be found in any textbook in probability theory (e.g., [5]).

**Exercise 5.26.** *Prove the following. Let* $\{X_n\}_{n\in\mathbb{N}}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}[X_i] = \mu$ *and* $\mathrm{Var}[X_i] = \sigma^2$*. Let* $S_n$ *be the sample average of the first n random variables. Then* $\sigma^{-1}\sqrt{n}(S_n - \mu)$ *converges in distribution to a standard normal random variable. HINT: make a change of variables and apply Theorem 5.10.*

### 5.5.3 Brownian motion

Let $\Omega$ be a sample space and $T > 0$. Recall that a stochastic process is a one parameter family $\{X(t)\}_{t\in[0,T]}$ of random variables $X(t) : \Omega \to \mathbb{R}$. We denote by $X(t,\omega)$ the value attained by the random variable $X(t)$ on the sample point $\omega \in \Omega$ (i.e., $X(t,\omega) = X(t)(\omega)$). Moreover, for each $\omega \in \Omega$ *fixed*, the function $t \to X(t,\omega)$ defined on $[0,T]$ is called a path of the stochastic process. In the following we refer to the parameter $t$ as the time variable, since this is what it represents in the applications that we have in mind.

**Definition 5.18.** *A Brownian motion, or Wiener process, is a stochastic process* $\{W(t)\}_{t\in[0,T]}$ *with the following properties:*

1. *For all*[6] $\omega \in \Omega$*, the paths are continuous (i.e.,* $t \to W(t,\omega)$ *is a continuous function on* $[0,T]$*) and* $W(0,\omega) = 0$*;*
2. *For all* $0 = t_0 < t_1 < \cdots < t_m = T$*, the increments*
$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1), \dots, W(T) - W(t_{m-1})$$
*are independent random variables and*
$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0, \quad \mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i, \quad \textit{for all } i = 0, \dots m-1;$$
3. *The increments are normally distributed, that is to say, for all* $0 \leq s < t \leq T$*,*
$$\mathbb{P}(W(t) - W(s) \in A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{y^2}{2(t-s)}}\, dy,$$
*for all regular sets* $A \subseteq \mathbb{R}$*.*

It can be shown that Brownian motions exist, yet a formal construction is technically quite difficult and beyond the purpose of these notes. One particular way to construct a Brownian motion is suggested in Exercise 5.17, namely by running a properly rescaled symmetric random walk for infinitely many steps. In fact it is useful to think of Brownian motions as time-continuum generalizations of the symmetric random walk. Note to this regard that the increments of a symmetric random walk also satisfy the independence property 2 in Definition 5.18. The fact that in the continuum limit they are normally distributed follows by the Central Limit Theorem.

[6] More precisely, for all $\omega \in A \subseteq \Omega$ with $\mathbb{P}(A) = 1$.

**Exercise 5.27 (•).** *Let* $\{W(t)\}_{t\in[0,T]}$ *be a Brownian motion. Show that* $\mathrm{Cov}[W(s), W(t)] = \min(s,t)$*, for all* $s,t \in [0,T]$*. (Compare this with Exercise 5.17.)*

# Chapter 6

# Black-Scholes options pricing theory

This final chapter is concerned with the most famous of all models in options pricing theory, namely the Black-Scholes model, which first appeared in the seminal paper [1]. The way we introduce it in this chapter is very different from the original argument in [1]. Our strategy is to derive the Black-Scholes model from the binomial options pricing model in the time-continuum limit. More precisely we let the number of steps $N$ in the binomial model goes to infinity and, at the same time, we let the length $h$ of each time step tends to zero, keeping constant the time of maturity $T = Nh$.

## 6.1 Black-Scholes markets

Can we use a Brownian motion to model the evolution in time of a stock price, i.e., can we set $S(t) = W(t)$? The answer is of course *no*, because $W(t)$ can also take negative values. In fact, since $W(t)$ is normally distributed, then it has a positive probability to take value in any interval within the real line. However this deficiency is easily corrected.

**Definition 6.1.** *Let $\{W(t)\}_{t\in[0,T]}$ be a Brownian motion, $\alpha \in \mathbb{R}$ and $\sigma > 0$. The positive stochastic process $\{S(t)\}_{t\in[0,T]}$ given by*

$$S(t) = S(0)e^{\alpha t+\sigma W(t)} \tag{6.1}$$

*is called a* **geometric Brownian motion**.

As the notation used in (6.1) suggests, we shall use geometric Brownian motions to model the dynamics of stock prices in the time-continuum case. More precisely, a **Black-Scholes market** is a market that consists of a risky asset, say a stock, whose price at time $t \in [0, T]$ is given by the geometric Brownian motion (6.1), and a risk-free asset with constant interest rate $r$; in particular, the value of the risk-free asset at time $t$ is given by $B(t) = B(0)e^{rt}$. Within this application, we call $\alpha$ the **instantaneous mean of log-return**, $\sigma$ the

**instantaneous volatility** and $\sigma^2$ the **instantaneous variance** of the geometric Brownian motion. To justify this terminology we now show that $\alpha$ and $\sigma$ satisfy the analogues of (5.16) in the time-continuum case, namely, for all $t \in [0, T]$ and $\varepsilon > 0$ such that $t + \varepsilon \leq T$ we have

$$\alpha = \frac{1}{\varepsilon}\mathbb{E}[\log S(t+\varepsilon) - \log S(t)], \quad \sigma^2 = \frac{1}{\varepsilon}\mathrm{Var}[\log S(t+\varepsilon) - \log S(t)]. \tag{6.2}$$

In fact, by (ii) of Theorem 5.6, and since $W(t) \in \mathcal{N}(0, t)$,

$$\begin{aligned}\mathbb{E}[\log S(t+\varepsilon) - \log S(t)] &= \mathbb{E}[\alpha\varepsilon + \sigma W(t+\varepsilon) - \sigma W(t)] \\ &= \alpha\varepsilon + \sigma(\mathbb{E}[W(t+\varepsilon)] - \mathbb{E}[W(t)]) = \alpha\varepsilon.\end{aligned}$$

The proof of the identity for $\sigma^2$ is left as an exercise.

**Exercise 6.1.** *Prove the second identity in* (6.2)*.*

In the following theorem we show that the historical variance (1.6) of a stock is an **unbiased estimator** for the instantaneous variance of the geometric Brownian motion used to model its price.

**Theorem 6.1.** *Suppose that at time $t = 0$ it is assumed that the stock price is described by a geometric Brownian motion in the interval $[0, T]$:*

$$S(t) = S(0)e^{\alpha t + \sigma W(t)}, \quad t \in [0, T].$$

*Given any arbitrary subinterval $[t_0, t] \subset [0, T]$ with length $\tau = t - t_0$, define the random variable*

$$\sigma_\tau^2(t) = \frac{1}{h(n-1)}\sum_{i=1}^{n}(R_i - \bar{R})^2, \tag{6.3}$$

*where $t_0 < t_1 < t_2 < \cdots < t_n = t$ is a partition of $[t_0, t]$ with $h = t_i - t_{i-1}$ and $R_i, \bar{R}$ are given by* (1.4)*,* (1.5)*. Then*

$$\mathbb{E}[\sigma_\tau(t)^2] = \sigma^2.$$

*In other words, $\sigma^2$ is the expected value of the $\tau$-historical variance at any time $t \in [0, T]$.*

*Proof.* In (6.3) we replace

$$\bar{R} = \frac{1}{n}\log\frac{S(t)}{S(t_0)} = \frac{1}{n}(\alpha(t - t_0) + \sigma(W(t) - W(t_0)) = \alpha h + \frac{\sigma}{n}(W(t) - W(t_0)), \text{ and}$$

$$R_i = \log\frac{S(t_i)}{S(t_{i-1})} = \alpha(t_i - t_{i-1}) + \sigma(W(t_i) - W(t_{i-1}) = \alpha h + \sigma(W(t_i) - W(t_{i-1}).$$

So doing we obtain

$$\begin{aligned}\sigma_\tau^2(t) &= \frac{\sigma^2}{h(n-1)}\sum_{i=1}^{n}\left[W(t_i) - W(t_{i-1}) - \frac{1}{n}(W(t) - W(t_0))\right]^2 \\ &= \frac{\sigma^2}{h(n-1)}\left[\sum_{i=1}^{n}(W(t_i) - W(t_{i-1}))^2 - \frac{1}{n}(W(t) - W(t_0))^2\right].\end{aligned}$$

Taking the expectation and using that $\mathbb{E}[(W(t)-W(s))^2] = \mathrm{Var}[W(t)-W(s)] = t-s$ we obtain

$$\mathbb{E}[\sigma_\tau^2] = \frac{\sigma^2}{h(n-1)}\left[\sum_{i=1}^n (t_i - t_{i-1}) - \frac{1}{n}(t-t_0)\right] = \frac{\sigma^2}{h(n-1)}(t-t_0)\left(1-\frac{1}{n}\right) = \sigma^2.$$

□

Next we derive the density function of the geometric Brownian motion.

**Theorem 6.2.** *The density of the random variable $S(t)$ is given by*

$$f_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right), \tag{6.4}$$

*where $H(x)$ is the* **Heaviside** *function, i.e.,*

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}.$$

*Proof.* The density of $S(t)$ is given by

$$f_{S(t)}(x) = \frac{d}{dx}F_{S(t)}(x),$$

where $F_{S(t)}$ is the distribution of $S(t)$, i.e.,

$$F_{S(t)}(x) = \mathbb{P}(S(t) \le x).$$

Clearly, $f_{S(t)}(x) = F_{S(t)}(x) = 0$, for $x \le 0$. For $x > 0$ we use that

$$S(t) \le x \quad \text{if and only if} \quad W(t) \le \frac{1}{\sigma}\left(\log\frac{x}{S(0)} - \alpha t\right) := A(x).$$

Thus

$$\mathbb{P}(S(t) \le x) = \mathbb{P}(-\infty < W(t) \le A(x)) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\,dy,$$

where for the second equality we used that $W(t) \in \mathcal{N}(0,t)$. Hence

$$f_{S(t)}(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{A(x)} e^{-\frac{y^2}{2t}}\,dy\right) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{A(x)^2}{2t}}\frac{dA(x)}{dx},$$

for $x > 0$, that is

$$f_{S(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left\{-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right\}, \quad x > 0.$$

The proof is complete. □

**Exercise 6.2.** *Express $\mathbb{P}(a < S(t) < b)$ in terms of the standard normal distribution $\Phi(x)$.*

Finally we study the relation between the discrete binomial stock price and the time-continuous geometric Brownian motion model. Consider an interval of time $[0, t]$ and a partition

$$0 = t_0 < t_1 < \cdots < t_N = t, \quad t_{i+1} - t_i = h > 0, \quad \text{for all } i = 0, \dots, N-1.$$

We define a binomial stock price on this partition by setting, for $u > d$,

$$S(t_i) = \begin{cases} S(t_{i-1})e^u & \text{with probability } p \\ S(t_{i-1})e^d & \text{with probability } 1-p \end{cases}.$$

Now consider $N$ independent and identically distributed random variables $X_1, \dots X_N$ satisfying

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}.$$

We can rewrite the binomial model above as

$$S(t_i) = S(t_{i-1}) \exp\left[\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) X_i\right].$$

Let us assume for simplicity that $p = 1/2$ (see Exercise 6.3 below for the general case). Then iterating the previous identity we obtain

$$S(t) = S(0) \exp\left[N\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_N\right],$$

where $M_N = X_1 + \dots X_N$ is the symmetric random walk stochastic process. Substituting $N = \frac{t}{h}$, the previous expression becomes

$$S(t) = S(0) \exp\left[\alpha t + \sigma\sqrt{h} M_{t/h}\right], \tag{6.5}$$

where $\alpha = (u+d)/(2h)$ and $\sigma = (u-d)/(2\sqrt{h})$ are the instantaneous mean of log-return and the instantaneous volatility of the binomial stock price, see (2.19). As indicated in Exercise 5.17, we have

$$\sqrt{h} M_{t/h} \approx W(t), \quad \text{for } h \text{ small}.$$

Therefore for small $h$ we can approximate (6.5) by the geometric Brownian motion

$$S(t) \approx S(0) \exp\left[\alpha t + \sigma W(t)\right].$$

The following simple application of the Central Limit Theorem turns the above argument into a rigorous result.

**Theorem 6.3.** *Consider the binomial stock price stochastic process* $\{S_N(t)\}_{t\in[0,T]}$ *given by*

$$S_N(t) = S(0)e^{\alpha t+\sigma\sqrt{t}\frac{M_N}{\sqrt{N}}},$$

*where* $M_N$ *is the symmetric random walk. Then*

$$S_N(t) \to S(t), \quad \text{in distribution as } N \to \infty,$$

*where* $S(t) = S(0)e^{\alpha t+\sigma W(t)}$ *is the geometric Brownian motion with instantaneous mean of log-return* $\alpha$ *and instantaneous volatility* $\sigma$.

*Proof.* We have, for all $x > 0$,

$$F_{S_N(t)}(x) = \mathbb{P}(S_N(t) \leq x) = \mathbb{P}(\frac{M_N}{\sqrt{N}} \leq \frac{\log\frac{x}{S(0)} - \alpha t}{\sigma\sqrt{t}}).$$

By the Central Limit Theorem, the random variable $\frac{M_N}{\sqrt{N}}$ converges in distribution to a random variable $X \in \mathcal{N}(0,1)$ as $N \to \infty$. Letting $X = W(t)/\sqrt{t}$ we then have

$$F_{S_N(t)}(x) \to \mathbb{P}(\frac{W(t)}{\sqrt{t}} \leq \frac{\log\frac{x}{S(0)} - \alpha t}{\sigma\sqrt{t}}) = \mathbb{P}(S(t) \leq x) = F_{S(t)}(x), \quad \text{as } N \to \infty.$$

□

**Exercise 6.3.** *Show that the binomial stock price converges in distribution to the geometric Brownian motion even for* $p \neq 1/2$. *Remark: it can be shown that the value of* $p \in (0,1)$ *only affects the rate of convergence. In particular the fastest convergence is obtained for* $p = 1/2$, *which is the reason why we always make this choice in the binomial model.*

**Exercise 6.4 (Matlab).** *Test numerically the convergence of the binomial stock to the geometric Brownian motion. Show that the fastest convergence is obtained for* $p = 1/2$.

## 6.2 Black-Scholes price of standard European derivatives

The ultimate purpose of this section is to introduce and justify the definition of fair price for standard European derivatives on Black-Scholes markets.

In Section 5.4 we showed that the binomial fair price at time $t = 0$ of the standard European derivative with pay-off $Y = g(S(N))$ at maturity $T = N$ can be written as $\Pi_Y(0) = e^{-rN}\mathbb{E}_q[g(S(N))]$, where $\mathbb{E}_q[\cdot]$ denotes the expectation in the risk-neutral probability measure; see (5.35). As a first step to derive the Black-Scholes price of the same European derivative, we first rewrite $\Pi_Y(0)$ in terms of the distribution of $S(N)$. Using the definition of probability measure in the $N$-coin toss probability space we have

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[g(S(N))] = e^{-rN}\sum_{\omega\in\Omega_N} q^{N_H(\omega)}(1-q)^{N_T(\omega)}g(S(N))$$
$$= e^{-rN}\sum_{k=0}^{N}\binom{N}{k}q^k(1-q)^{N-k}g(s_k),$$

where $s_0,\dots,s_N$ are the possible values of $S(N)$, i.e.,

$$s_k = S(0)\exp(ku+(N-k)d), \quad k = 0\dots,N, \tag{6.6}$$

and

$$q = \frac{e^r - e^d}{e^u - e^d}, \quad r \in (d,u)\ (\Leftrightarrow q\in(0,1)). \tag{6.7}$$

Note the important formula

$$k = \frac{1}{u-d}\left(\log\left(\frac{s_k}{S(0)}\right) - Nd\right), \tag{6.8}$$

which is obtained by inverting (6.6). We can rewrite $\Pi_Y(0)$ as

$$\Pi_Y(0) = e^{-rN}\sum_{k=0}^{N}\left(\frac{q}{p}\right)^k\left(\frac{1-q}{1-p}\right)^{N-k}\binom{N}{k}p^k(1-p)^{N-k}g(s_k).$$

Recalling that the distribution of $S(N)$ is given by $\binom{N}{k}p^k(1-p)^{N-k}$ (see Section 5.4), we obtain

$$\Pi_Y(0) = e^{-rN}\sum_{k=0}^{N} Z_N(s_k)f_{S(N)}(s_k)g(s_k), \tag{6.9}$$

where

$$Z_N(s_k) = \left(\frac{q}{p}\right)^k\left(\frac{1-q}{1-p}\right)^{N-k} = \left(\frac{q}{p}\right)^{\frac{1}{u-d}\left[\log\left(\frac{s_k}{S(0)}\right)-Nd\right]}\left(\frac{1-q}{1-p}\right)^{N-\frac{1}{u-d}\left[\log\left(\frac{s_k}{S(0)}\right)-Nd\right]}, \tag{6.10}$$

the second equality following by (6.8). From now on we assume $p = 1/2$ for simplicity. Denoting $s_k = x$, we rewrite (6.9) as

$$\Pi_Y(0) = e^{-rN}\sum_{x\in \mathrm{Im}S(N)} Z_N(x)f_{S(N)}(x)g(x), \tag{6.11}$$

where $\mathrm{Im}S(N)$ is the image of $S(N)$, $f_{S(N)}$ is the probability distribution of $S(N)$, and

$$Z_N(x) = (2q)^{\frac{1}{u-d}\left[\log\left(\frac{x}{S(0)}\right)-Nd\right]}(2(1-q))^{N-\frac{1}{u-d}\left[\log\left(\frac{x}{S(0)}\right)-Nd\right]}. \tag{6.12}$$

We have also seen in Section 6.1 that, in a suitable limit, the binomial stock price converges to the geometric Brownian motion, see Theorem 6.3. We shall obtain the Black-Scholes formula in the same limit applied to (6.11). To this purpose, let $T > 0$ and consider the partition of the interval $[0, T]$ given by

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_j - t_{j-1} = h > 0, \quad j \in \mathcal{I}.$$

Let us define the binomial model $S(0) = S_0 > 0$,

$$S(t_j) = \begin{cases} S(t_{j-1})e^u, & \text{with probability } p = 1/2 \\ S(t_{j-1})e^d, & \text{with probability } p = 1/2 \end{cases}, \quad j \in \mathcal{I}, \tag{6.13}$$

where

$$u = \alpha h + \sigma\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{h}. \tag{6.14}$$

Note that

$$\sigma = \frac{u-d}{2\sqrt{h}}, \quad \alpha = \frac{u+d}{2h},$$

hence, by Theorem 6.3, in the limit $h \to 0$ the price of the stock follows the geometric Brownian motion $S(t) = S(0)\exp(\alpha t + \sigma W(t))$. Moreover $B(t_j) = B_0 e^{rt_j} = B_0 e^{rhj}$. Hence the pair $(\widetilde{S}(j) = S(t_j),\ \widetilde{B}(j) = B(t_j))$ is equivalent to a binomial market with parameters $(u, d, rh)$ and $p = 1/2$. In particular the parameter $q$ defined by (6.7) becomes

$$q = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}} \tag{6.15}$$

and since

$$\alpha h - \sigma\sqrt{h} < rh < \alpha h + \sigma\sqrt{h}$$

holds for $h$ small, then we can assume that $q \in (0, 1)$ and that the market is arbitrage-free. Therefore the initial price of the European derivative with pay-off $Y = g(\widetilde{S}(N)) = g(S(T))$ is given by (6.11). Using $Nh = T$, we rewrite (6.11) as

$$\Pi_Y(0) = e^{-rhN} \sum_{x \in \mathrm{Im}\widetilde{S}(N)} Z_N(x) f_{\widetilde{S}(N)}(x) g(x) = e^{-rT} \sum_{x \in \mathrm{Im}S(T)} Q_h(x) f_{S(T)}(x) g(x), \tag{6.16}$$

where $Q_h(x) = Z_{T/h}(x)$; by (6.12), (6.15) and the definitions of $u, d$, we have $Q_h(x) = \eta_h(x)\xi_h(x)$, where

$$\eta_h(x) = \left(2\frac{e^{rh} - e^{\alpha h - \sigma\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}}\right)^{\frac{1}{2\sigma\sqrt{h}}\left(\log\frac{x}{S(0)} - \alpha T\right) + \frac{T}{2h}},$$

$$\xi_h(x) = \left(2\frac{e^{\alpha h + \sigma\sqrt{h}} - e^{rh}}{e^{\alpha h + \sigma\sqrt{h}} - e^{\alpha h - \sigma\sqrt{h}}}\right)^{-\frac{1}{2\sigma\sqrt{h}}\left(\log\frac{x}{S(0)} - \alpha T\right) + \frac{T}{2h}}.$$

**Theorem 6.4.** *The following holds:*

$$\lim_{h\to 0} Q_h(x) = e^{-\frac{(\alpha+\frac{\sigma^2}{2}-r)^2 T}{2\sigma^2}} e^{-\frac{1}{\sigma^2}\left(\log\frac{x}{S(0)}-\alpha T\right)\left(\alpha+\frac{\sigma^2}{2}-r\right)} := Q(x).$$

**Exercise 6.5.** *Prove the theorem.*

We are now in the position to justify the definition of the Black-Scholes price of the derivative. In view of the analogies between finite and uncountable probability spaces pointed out in Theorem 5.6, the natural generalization of (6.16) in the time-continuum case is

$$\Pi_Y(0) = e^{-rT}\int_{\mathbb{R}} Q(x) f_{S(T)}(x) g(x)\,dx,$$

where $f_{S(T)}$ is now the probability density of the random variable $S(T) = S(0)e^{\alpha T+\sigma W(T)}$, i.e.,

$$f_{S(T)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 T}}\frac{1}{x}\exp\left(-\frac{(\log\frac{x}{S(0)}-\alpha T)^2}{2\sigma^2 T}\right), \tag{6.17}$$

see (6.4). Explicitly, $\Pi_Y(0)$ is given by

$$\Pi_Y(0) = \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_0^\infty e^{-\frac{(\alpha+\frac{\sigma^2}{2}-r)^2 T}{2\sigma^2}} e^{-\frac{1}{\sigma^2}(\log\frac{x}{S(0)}-\alpha T)(\alpha+\frac{\sigma^2}{2}-r)} e^{-\frac{\left(\log\frac{x}{S(0)}-\alpha T\right)^2}{2\sigma^2 T}} \frac{g(x)}{x}\,dx.$$

Now, the exponents in the exponential functions inside the integral form a perfect square:

$$\begin{aligned}\Pi_Y(0) &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_0^\infty e^{-\frac{1}{2\sigma^2 T}\left[\left(\log\frac{x}{S(0)}-\alpha T\right)+\left(\alpha+\frac{\sigma^2}{2}-r\right)T\right]^2}\frac{g(x)}{x}\,dx\\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_0^\infty e^{-\frac{1}{2\sigma^2 T}\left[\log\frac{x}{S(0)}+\left(\frac{\sigma^2}{2}-r\right)T\right]^2}\frac{g(x)}{x}\,dx.\end{aligned}$$

Note that the dependence on $\alpha$ is gone! Finally, the change of variable

$$y = \frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{S(0)}+\left(\frac{\sigma^2}{2}-r\right)T\right)$$

gives

$$\Pi_Y(0) = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{\mathbb{R}} g\left(S(0)e^{(r-\frac{\sigma^2}{2})T}e^{\sigma\sqrt{T}y}\right)e^{-\frac{y^2}{2}}\,dy.$$

The Black-Scholes price $\Pi_Y(t_0)$ at a generic time $t_0\in[0,T]$ is obtained by assuming that the present time is $t=t_0$ (instead of $t=0$) and replacing the expiration date $T$ with the time left to maturity, i.e., $\tau = T-t_0$. The latter is of course quite reasonable and can be justified by the same argument applied to derive the formula for $\Pi_Y(0)$. We are led to introduce the following definition.

**Definition 6.2.** *Consider the European derivative with pay-off $Y = g(S(T))$ and time of maturity $T > 0$. Let $r$ be the (constant) interest rate of the money market and assume that the price of the stock is given by the geometric Brownian motion $S(t) = S(0)e^{\alpha t+\sigma W(t)}$, $t \in [0,T]$. The Black-Scholes price $\Pi_Y(t)$ of the derivative at time $t \in [0,T]$ is defined as*

$$\Pi_Y(t) = v(t, S(t)), \tag{6.18}$$

*where*

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\, y}\right) e^{-\frac{y^2}{2}}\, dy, \quad \tau = T-t, \tag{6.19}$$

*is called the* **Black-Scholes price function** *of the derivative.*

**Remark 6.1.** Of course we are tacitly assuming that the pay-off function $g$ is such that the integral in the right hand side of (6.19) is well-defined. Note also that the Black-Scholes price at time $t$ is a deterministic function of $S(t)$ and thus can be computed with the information available at time $t$.

**Remark 6.2.** The fact that the Black-Scholes price is independent of the mean of log-return $\alpha$ of the stock is consistent with the numerical observation made in Section 3.3 that the binomial price of European options is weakly dependent on the parameter $\alpha$. In the time-continuum limit, this dependence disappears completely.

Note that the definition (6.18)-(6.19) of the Black-Scholes price can be written also in the probabilistic form

$$\Pi_Y(t) = e^{-r\tau}\mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(W(T)-W(t))}]. \tag{6.20}$$

In fact, since $G = (W(T)-W(t))/\sqrt{T-t} \in \mathcal{N}(0,1)$, then

$$\begin{aligned}\mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(W(T)-W(t))}] &= \mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G})] \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(S(t)e^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\, y}\right) e^{-\frac{y^2}{2}}\, dy,\end{aligned}$$

where we used Theorem 5.6.

## Hedging portfolio

A portfolio process $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ invested in the Black-Scholes market is said to be hedging the European derivative with pay-off $Y$ and maturity $T > 0$ if $V(T) = Y$, where $V(t) = h_S(t)S(t) + h_B(t)B(t)$ is the value of the portfolio process at time $t \in [0,T]$. The portfolio process is said to be replicating the derivative if $V(t) = \Pi_Y(t)$, where $\Pi_Y(t)$ is the Black-Scholes price of the derivative. It can be shown that the Black-Scholes $\Pi_Y(t)$ coincides with the value at time $t \in [0,T]$ of any self-financing portfolio processes hedging the derivative. Thus Definition 6.2 can be motivated by the same argument used to justify the definition of binomial price of European derivatives, see Remark 3.1. However this approach requires the use of stochastic calculus and it is therefore beyond the purpose of these notes. Moreover it can be shown that the portfolio process $\{(h_S(t), h_B(t))\}_{t\in[0,T]}$ given by

$$h_S(t) = \Delta(t, S(t)), \quad \Delta(t,x) = \partial_x v(t,x), \quad h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) \tag{6.21}$$

is self-financing and hedges the derivative. This portfolio is also predictable, as the position at time $t$ depends only on the stock price $S(t)$. For a heuristic derivation of (6.21), see Exercise 6.6 below. In the next two sections we compute the Black-Scholes price and the hedging portfolio process of some simple derivatives.

**Exercise 6.6.** *The goal of this exercise is to justify the formula* (6.21) *for a self-financing and hedging portfolio process of standard European derivatives priced by the Black-Scholes formula. Recall that in the time discrete case we have*

$$h_S(t) = \frac{1}{S(t-1)} \frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}, \tag{6.22}$$

*see Theorem 3.3. Now, as for the derivation of the Black-Scholes price given in this section, consider a partition* $\{t_0, \dots, t_N\}$ *of the interval* $[0,T]$ *and let*

$$t-1 = t_{j-1},\ t = t_j = t_{j-1} + h,\ u = \alpha h + \sigma\sqrt{h},\ d = \alpha h - \sigma\sqrt{h},$$

*see* (6.13) *and* (6.14). *Let* $\Pi_Y(t_j) = v(t_j, S(t_j))$, *where* $v$ *is the Black-Scholes price function. Show that* (6.22) *converges to* $\partial_x v(t,x)$ *as* $h \to 0$. *Prove that* (6.21) *replicates the derivative.*

## 6.3 Black-Scholes price of European call and put options

In this section we focus the discussion on call/put options. We thereby assume that the pay-off of the derivative is given by

$$Y = (S(T) - K)_+, \text{ i.e., } Y = g(S(T)),\ g(z) = (z - K)_+, \quad \text{for a call option,}$$

$$Y = (K - S(T))_+, \text{ i.e., } Y = g(S(T)),\ g(z) = (K - z)_+, \quad \text{for a put option.}$$

The function $v$ given by (6.19) will be denoted by $C$, for a call option, and by $P$, for a put option.

**Theorem 6.5.** *The Black-Scholes price at time $t$ of the European call option with strike price $K > 0$ and maturity $T > 0$ is given by $C(t, S(t))$, where*

$$C(t,x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{6.23a}$$

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{6.23b}$$

*and where* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$ *is the standard normal distribution. The Black-Scholes price of the corresponding put option is given by* $P(t, S(t))$*, where*

$$P(t,x) = \Phi(-d_2)Ke^{-r\tau} - \Phi(-d_1)x. \tag{6.24}$$

*Moreover the* **put-call parity identity** *holds:*

$$C(t, S(t)) - P(t, S(t)) = S(t) - Ke^{-r\tau}. \tag{6.25}$$

*Proof.* We derive the Black-Scholes price of call options only, the argument for put options being similar (see Exercise 6.7 below). We substitute $g(z) = (z - K)_+$ into the right hand side of (6.19) and obtain

$$C(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} - K \right)_+ e^{-\frac{y^2}{2}}\, dy.$$

Now we use that $xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} > K$ if and only if $y > -d_2$. Hence

$$C(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[ \int_{-d_2}^{\infty} xe^{(r-\frac{1}{2}\sigma^2)\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}} - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right].$$

Using $-\frac{1}{2}y^2 + \sigma\sqrt{\tau}y = -\frac{1}{2}(y - \sigma\sqrt{\tau})^2 + \frac{\sigma^2}{2}\tau$ and changing variable in the integrals we obtain

$$\begin{aligned}
C(t,x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[ xe^{r\tau} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\, dy - K \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy \right] \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} \left[ xe^{r\tau} \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} e^{-\frac{1}{2}y^2}\, dy - K \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\, dy \right] \\
&= x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2).
\end{aligned}$$

As to the put-call parity, we have

$$\begin{aligned}
C(t,x) - P(t,x) &= x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) - \Phi(-d_2)Ke^{-r\tau} + x\Phi(-d_1) \\
&= x(\Phi(d_1) + \Phi(-d_1)) - Ke^{-r\tau}(\Phi(d_2) + \Phi(-d_2)).
\end{aligned}$$

As $\Phi(z) + \Phi(-z) = 1$, the claim follows. □

**Remark 6.3.** Note that the Black-Scholes price of call and put options is strictly positive for all $t \in [0, T)$, while the binomial price can also attain the (unrealistic) zero value prior to maturity, see Section 3.3.

**Exercise 6.7.** *Derive the Black-Scholes price* $P(t, S(t))$ *of European put options claimed in Theorem 6.5.*

**Exercise 6.8 (Matlab).** *Write a Matlab function that computes the Black-Scholes price of call/put options. Show numerically that the binomial price of call/put options converges to their Black-Scholes price in the time continuum limit.*

**Exercise 6.9** (•). *Prove that*

$$\lim_{\sigma\to 0^+} C(t,x) = (x - Ke^{-r\tau})_+, \quad \lim_{\sigma\to\infty} C(t,x) = x.$$

*Compute also the following limits:*

$$\lim_{K\to 0^+} C(t,x), \qquad \lim_{K\to +\infty} C(t,x), \qquad \lim_{T\to +\infty} C(t,x), \qquad \lim_{x\to 0^+} C(t,x).$$

*Repeat all the above for put options.*

Next we derive the hedging portfolio for call and put options.

**Theorem 6.6.** *The following portfolio processes are self-financing hedging portfolios for European call/put options on Black-Scholes markets:*

$$h_S(t) = \Phi(d_1), \quad h_B(t) = -\frac{Ke^{-r\tau}\Phi(d_2)}{B(t)} \quad \textit{for call options} \tag{6.26a}$$

$$h_S(t) = -\Phi(-d_1), \quad h_B(t) = \frac{Ke^{-r\tau}\Phi(-d_2)}{B(t)} \quad \textit{for put options.} \tag{6.26b}$$

*Proof.* Recall that $h_S(t) = \partial_x C(t, S(t))$ for call options and $h_S(t) = \partial_x P(t, S(t))$ for put options, see (6.21), while the number of shares of the risk-free asset in the hedging portfolio is given by

$$h_B(t) = (C(t,S(t)) - S(t)\partial_x C(t,S(t)))/B(t), \quad \text{for call options,} \tag{6.27a}$$

$$h_B(t) = (P(t,S(t)) - S(t)\partial_x P(t,S(t)))/B(t), \quad \text{for put options.} \tag{6.27b}$$

Let us compute $\partial_x C$:

$$\partial_x C = \Phi(d_1) + x\Phi'(d_1)\partial_x d_1 - Ke^{-r\tau}\Phi'(d_2)\partial_x d_2.$$

As $\partial_x d_1 = \partial_x d_2 = \frac{1}{\sigma\sqrt{\tau}x}$, and $\Phi'(x) = e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, we obtain

$$\partial_x C = \Phi(d_1) + \frac{1}{\sigma\sqrt{2\pi\tau}}\left(e^{-\frac{1}{2}d_1^2} - \frac{K}{x}e^{-r\tau}e^{-\frac{1}{2}d_2^2}\right).$$

Replacing $d_1 = d_2 + \sigma\sqrt{\tau}$ we obtain

$$\partial_x C = \Phi(d_1) + \frac{e^{-\frac{1}{2}d_2^2}}{\sigma\sqrt{2\pi\tau}}\left(e^{-\frac{1}{2}\sigma^2\tau - d_2\sigma\sqrt{\tau}} - \frac{K}{x}e^{-r\tau}\right).$$

Using the definition of $d_2$, the term within round brackets in the previous expression is easily found to be zero, hence

$$\partial_x C = \Phi(d_1).$$

By the put-call parity we find also

$$\partial_x P = \Phi(d_1) - 1 = -\Phi(-d_1).$$

In both cases, the number of shares in the risk-free asset is computed using (6.27). □

**Remark 6.4.** Note that $\partial_x C > 0$, while $\partial_x P < 0$. This agrees with the fact that call options are bought to protect a short position on the underlying stock, while put options are bought to protect a long position on the underlying stock.

### The greeks

The Black-Scholes price of a call (or put) option derived in Theorem 6.5 depends on the price of the underlying stock, the time to maturity, the strike price, as well as on the (constant) market parameters $r, \sigma$ (it does not depend on $\alpha$). The partial derivatives of the price function $C$ with respect to these variables are called **greeks**. We collect the most important ones (for call options) in the following theorem.

**Theorem 6.7.** *The price function $C$ of call options satisfies the following:*

$$\Delta := \partial_x C = \Phi(d_1), \tag{6.28}$$

$$\Gamma := \partial_x^2 C = \frac{\phi(d_1)}{x\sigma\sqrt{\tau}}, \tag{6.29}$$

$$\rho := \partial_r C = K\tau e^{-r\tau}\Phi(d_2), \tag{6.30}$$

$$\Theta := \partial_t C = -\frac{x\phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2), \tag{6.31}$$

$$\nu := \partial_\sigma C = x\phi(d_1)\sqrt{\tau} \quad \textit{(called “vega”)}, \tag{6.32}$$

*where $\phi(z) = \Phi'(z) = (\sqrt{2\pi})^{-1}e^{-\frac{z^2}{2}}$. In particular:*

- $\Delta > 0$, *i.e., the price of a call is increasing on the price of the underlying stock;*
- $\Gamma > 0$, *i.e., the price of a call is convex on the price of the underlying stock;*
- $\rho > 0$, *i.e., the price of the call is increasing on the interest rate of the risk-free asset;*
- $\Theta < 0$, *i.e., the price of the call is decreasing in time;*
- $\nu > 0$, *i.e., the price of the call is increasing on the volatility of the stock.*

**Exercise 6.10.** *Use the put-call parity to derive the greeks of put options.*

The greeks measure the sensitivity of option prices with respect to the market conditions. This information can be used to draw some important conclusions. For instance, let us comment on the fact that vega is positive. It implies that the wish of an investor with a long position on a call option is that the volatility of the underlying stock increased. As usual, since this might not happen, the buyer of the call may incur in a loss if the stock volatility decreases (since the call option will loose value). This exposure to volatility can be secured by adding volatility swaps into the portfolio.

## 6.4 The Black-Scholes price of other standard European derivatives

In this section we present a few more applications of the Black-Scholes formula (6.18)-(6.19).

### Binary call option

A **binary** (or **digital**) call option with strike $K$ and maturity $T$ pays-off the buyer if and only if $S(T) > K$. If the pay-off is a fixed amount of cash $L$, then the binary call option is said to be "cash-settled", while if the pay-off is the stock itself then the option is said to be "physically settled". Let us compute the Black-Scholes price of the cash-settled binary call option. The pay-off is $Y = LH(S(T)-K)$, where $H(z)$ is the Heaviside function. Replacing $g(z) = LH(z-K)$ into (6.19) we obtain

$$\begin{aligned}
v(t,x) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} L \int_{\mathbb{R}} H(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\,y} - K) e^{-\frac{y^2}{2}}\, dy \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi}} L \int_{-d_2}^{\infty} e^{-\frac{y^2}{2}}\, dy,
\end{aligned}$$

where we recall that $d_2 = \frac{1}{\sigma\sqrt{\tau}}[\log(x/K) + (r - \frac{\sigma^2}{2})\tau]$. With the change of variable $y \to -y$ we obtain

$$\Pi_Y(t) = e^{-r\tau} L\Phi(d_2).$$

**Exercise 6.11** ($\star$). *Compute the Black-Scholes price of the physically-settled binary call.*

**Exercise 6.12.** *Compute the hedging portfolio of physically/cash-settled binary options.*

### Butterfly options strategy

Given $K, \Delta K > 0$, a **butterfly option strategy** is a constant portfolio which is

- Long 1 call with strike price $K - \Delta K$ and maturity $T$;
- Short 2 calls with strike price $K$ and maturity $T$;
- Long 1 call with strike price $K + \Delta K$ and maturity $T$.

The portfolio value at expire is

$$V(T) = (S(T) - K + \Delta K)_+ - 2(S(T) - K)_+ + (S(T) - K - \Delta K)_+.$$

We may think of $V(T)$ as the pay-off $Y$ of a standard European derivative, where $Y = g(S(T))$ and

$$g(x) = (x - K + \Delta K)_+ - 2(x - K)_+ + (x - K - \Delta K)_+ = g_1(x) + g_2(x) + g_3(x).$$

Note that the pay-off is positive, i.e., the option expires in the money, if and only if $S(T) \in (K - \Delta K, K + \Delta K)$. Hence a butterfly option strategy is lucrative when the price of the stock at time $T$ lies within an interval centered in $K$. As the integral in (6.19) is linear in the pay-off function $g$, the value of a butterfly portfolio at time $t$ is $V(t) = v(t, S(t))$, where

$$v(t,x) = v_1(t,x) + v_2(t,x) + v_3(t,x), \quad v_i(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g_i \left( x e^{(r - \frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}\, y} \right) e^{-\frac{y^2}{2}} \, dy.$$

As $g_1, g_2, g_3$ are pay-off functions of call options, we obtain

$$V(t) = \Pi_Y(t) = C(t, S(t), K - \Delta K, T) - 2C(t, S(t), K, T) + C(t, S(t), K + \Delta K, T),$$

where we denoted by $C(t, S(t), K, T)$ the Black-Scholes price of the call with strike $K$ and maturity $T$. This notation, and the analogous one $P(t, S(t), K, T)$ for put options, will be used in the rest of this section.

**Remark 6.5.** We obtain the same result by applying the dominance principle to the butterfly portfolio.

## Chooser option

This is an example of "second derivative", i.e., of a financial derivative whose underlying is another derivative. More precisely, given $T_2 > T_1$, a **chooser option** with maturity $T_1$ is a contract which gives to the buyer the right to choose at time $T_1$ whether the derivative becomes a call or a put option with strike $K$ and maturity $T_2$. Hence the pay-off at time $T_1$ for this derivative is

$$Y = \max(C(t_1, S(T_1), K, T_2), P(t_1, S(T_1), K, T_2)).$$

Using the identity $\max(a, b) = a + \max(0, b - a)$ we obtain

$$Y = C(t_1, S(T_1), K, T_2) + \max(0, P(t_1, S(T_1), K, T_2) - C(t_1, S(T_1), K, T_2)).$$

By the put-call parity,

$$Y = C(t_1, S(T_1), K, T_2) + \max(0, Ke^{-r(T_2 - T_1)} - S(T_1)) = Z + U. \tag{6.33}$$

Hence $\Pi_Y(t) = \Pi_Z(t) + \Pi_U(t)$. Since $U$ is the pay-off of a put option with strike $Ke^{-r(T_2 - T_1)}$ expiring at time $T_1$ then

$$\Pi_U(t) = P(t, S(t), Ke^{-r(T_2 - T_1)}, T_1). \tag{6.34}$$

To find $\Pi_Z(t)$ we need the following theorem:

**Theorem 6.8** $(*)$**.** *Consider the standard European derivative with pay-off $Y = g(S(T))$ at maturity $T$ and the derivative with pay-off $Z = \Pi_Y(t_*)$ at maturity $t_* < T$. Then $\Pi_Z(t) = \Pi_Y(t)$, $t \in [0, t_*]$.*

*Proof.* We have

$$Z = \Pi_Y(t_*) = e^{-r(T-t_*)} \int_{\mathbb{R}} g\left(S(t_*)e^{(r-\frac{\sigma^2}{2})(T-t_*)+\sigma\sqrt{T-t_*}y}\right) e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} := h(S(t_*)).$$

This shows that the derivative with pay-off $Z$ and maturity $t_*$ is a standard European derivative with pay-off function $h$. Hence, applying again Black-Scholes' formula, we obtain

$$\begin{aligned}
\Pi_Z(t) &= e^{-r(t_*-t)} \int_{\mathbb{R}} h\left(S(t)e^{(r-\frac{\sigma^2}{2})(t_*-t)+\sigma\sqrt{t_*-t}z}\right) e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}} \\
&= e^{-r(t_*-t)}e^{-r(T-t_*)} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} g\Big(S(t)e^{(r-\frac{\sigma^2}{2})(t_*-t)+\sigma\sqrt{t_*-t}z}e^{(r-\frac{\sigma^2}{2})(T-t_*)+\sigma\sqrt{T-t_*}y}\Big) \right. \\
&\quad \left. \times \exp\big(-\frac{1}{2}y^2 - \frac{1}{2}z^2\big)\frac{dy}{\sqrt{2\pi}}\right]\frac{dz}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}}\int_{\mathbb{R}} g\Big(S(t)e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\sqrt{t_*-t}z+\sqrt{T-t_*}y)}\Big)e^{-\frac{1}{2}(y^2+z^2)}\frac{d(y,z)}{2\pi} \\
&= e^{-r(T-t)}\mathbb{E}[g\Big(xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\sqrt{t_*-t}Z+\sqrt{T-t_*}Y)}\Big)]_{x=S(t)},
\end{aligned}$$

where $X, Y \in \mathcal{N}(0,1)$ are independent and so their joint distribution is given by $e^{-\frac{1}{2}y^2-\frac{1}{2}z^2}/2\pi$, see Theorem 5.7. Let $G_1 := \sqrt{t_*-t}\,Z$ and $G_2 := \sqrt{T-t_*}\,Z$. As $G_1 \in \mathcal{N}(0, t_*-t)$ and $G_2 \in \mathcal{N}(0, T-t_*)$, Theorem 5.9 gives $G_1 + G_2 \in \mathcal{N}(0, T-t)$, hence the above calculation leads to

$$\Pi_Z(t) = e^{-r(T-t)} \int_{\mathbb{R}} g\Big(S(t)e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}x}\Big)e^{-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}},$$

which is the claim. □

Applying Theorem 6.8 with $Z = C(t_1, S(T_1), K, T_2) = \Pi_{(S(T_2)-K)_+}(T_1)$ we obtain

$$\Pi_Z(t) = C(t, S(t), K, T_2). \tag{6.35}$$

Replacing (6.34) and (6.35) into (6.33) we finally obtain, for the Black-Scholes price of the chooser option,

$$\Pi_Y(t) = C(t, S(t), K, T_2) + P(t, S(t), Ke^{-r(T_2-T_1)}, T_1).$$

**Exercise 6.13 (•).** *Consider the European derivative with maturity $T$ and pay-off $Y$ given by*

$$Y = k + S(T)\log S(T),$$

*where $k > 0$ is a constant. Find the Black-Scholes price of the derivative at time $t < T$ and the self-financing hedging portfolio. Find the probability that the derivative expires in the money.*

**Exercise 6.14** (•). *Consider the European derivative with pay-off* $Y = S(T)(S(T) - K)$ *and time of maturity* $T$*, where* $K > 0$ *is a constant. Compute the Black-Scholes price* $\Pi_Y(t)$ *of this derivative and the self-financing hedging portfolio. Finally, assume* $S(0) = K$ *and compute the expected relative return of a constant portfolio with 1 share of this derivative.*

**Exercise 6.15** (•). *Let* $K > 0$*. A European style derivative on a stock with maturity* $T > 0$ *gives to its owner the right to choose between selling the stock for the price* $K$ *at time* $T$ *or paying the amount* $K$ *at time* $T$*. Draw the pay-off function of the derivative. Compute the Black-Scholes price of the derivative. Show that there exists a value* $K_* > 0$ *of* $K$ *such that the Black-Scholes price of the derivative is zero. What is the financial interpretation of* $K_*$*?.*

**Exercise 6.16** ($\star$). *Compute the Black-Scholes price* $\Pi_Y(0)$ *at time* $t = 0$ *of the European derivative with pay-off* $Y = \max(S(T), B(T))$*, where* $B(t)$ *is the price of the risk-free asset,* $S(t)$ *is the price of the underlying stock and* $T$ *is the time of maturity of the derivative. Derive the low volatility limit (*$\sigma \to 0^+$*) and the high volatility limit (*$\sigma \to +\infty$*) of* $\Pi_Y(0)$*.*

## 6.5 Implied volatility

Let $C(t, S(t), K, T)$ denote the Black-Scholes price of the European call with strike price $K$, maturity time $T$ on a stock with price $S(t)$ at time $t$. In fact, assuming that the underlying stock pays no dividend before time $T$, this is also the Black-Scholes price of the corresponding American call, which is the one most often traded in the market. Recall that in the derivation of the Black-Scholes price it is assumed that the price of the stock follows the geometric Brownian motion

$$S(t) = S(0)e^{\alpha t + \sigma W(t)},$$

where $\{W(t)\}_{t\in[0,T]}$ is a Brownian motion stochastic process, $\sigma > 0$ is the instantaneous volatility and $\alpha \in \mathbb{R}$ is the instantaneous mean of log-return. The function $C(t, x, K, T)$ is given by

$$C(t, x, K, T) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

where $r > 0$ is the (constant) interest rate of the money market, $\tau = T - t$ is the time left to the expiration of the call,

$$d_2 = \frac{\log \frac{x}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}$$

and $\Phi$ denotes the standard normal distribution,

$$\Phi(z) = \int_{-\infty}^{z} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}.$$

Remarkably, $C(t, S(t), K, T)$ does not depend on the mean of log-return $\alpha$ of the stock price. However it depends on the parameters $(\sigma, r)$ and since here we are particularly interested in the dependence on the volatility, we re-denote the Black-Scholes price of the call as

$$C(t, S(t), K, T, \sigma).$$

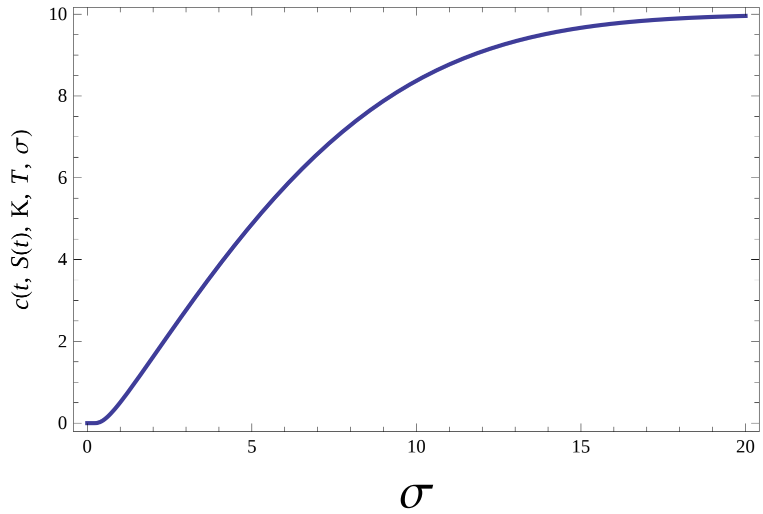



Figure 6.1: We fix $S(t) = 10$, $K = 12$, $r = 0.01$, $\tau = 1/12$ and depict the Black-Scholes price of the call as a function of the volatility. Note that in practice only the very left part of this picture is of interest, because typically $0 < \sigma < 1$.

Moreover, as shown in Theorem 6.7,

$$\frac{\partial C}{\partial \sigma} = \text{vega} = \frac{x}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} > 0.$$

Hence the Black-Scholes price of the option is an increasing function of the volatility. Furthermore,

$$\lim_{\sigma \to 0^+} C(t, S(t), K, T, \sigma) = (S(t) - Ke^{-r\tau})_+, \quad \lim_{\sigma \to +\infty} C(t, S(t), K, T, \sigma) = S(t),$$

see Exercise 6.9. Therefore the function $C(t, S(t), K, T, \cdot)$ is a one-to-one map from $(0, \infty)$ into the interval $I = ((S(t) - Ke^{-r\tau})_+, S(t))$, see Figure 6.1. Now suppose that at some given *fixed* time $t$ the real market price of the call is $\widetilde{C}(t)$. Clearly, the option is always cheaper than the stock (otherwise we would buy directly the stock, and not the option) and typically we also have $\widetilde{C}(t) > \max(0, S(t) - Ke^{-r\tau})$. The latter is always true if $S(t) < Ke^{-r\tau}$ (the market price of options is positive), while if $S(t) > Ke^{-r\tau}$ this follows by the fact that $S(t) - Ke^{-r\tau} \approx S(t) - K$ and real calls are always more expansive than their intrinsic value. This being said, we can safely assume that $\widetilde{C}(t) \in I$.

Thus given the value of $\widetilde{C}(t)$ there exists a unique value of $\sigma$, which we denote by $\sigma_{\text{imp}}$, such that

$$C(t, S(t), K, T, \sigma_{\text{imp}}) = \widetilde{C}(t).$$

$\sigma_{\text{imp}}$ is called the **implied volatility** of the option. The implied volatility must be computed numerically (for instance using Newton's method), since there is no close formula for it. Moreover it is usually computed using "nearly at the money" calls.

The implied volatility of an option (in this example of a call option) is a very important parameter and it is often quoted together with the price of the option. If the market followed exactly the assumptions in the Black-Scholes theory, then the implied volatility would be constant (independent of time) and it would be the same for all call options on the same stock with the same strike and maturity. However for real market options this turns out to be false, i.e., the implied volatility depends on time, $K$ and $T$. In this respect, $\sigma_{\text{imp}}$ may be viewed as a quantitative measure of how real markets deviate from ideal Black-Scholes markets. The implied volatility may also be viewed as the market consensus on the expected future value of the volatility of the underlying stock. That is to say, by pricing the option at the price $\widetilde{C}(t) = C(t, S(t), K, T, \sigma_{\text{imp}})$, the market participants are telling us that they believe that the volatility of the stock in the future will be $\sigma_{\text{imp}}$. (This of course provided one assumes the real markets are fair.)

### Volatility smile

As mentioned before, the implied volatility in real markets depends on the parameters $K, T$. Here we are particularly interested in the dependence on the strike price, hence we re-denote the implied volatility as $\sigma_{\text{imp}}(K)$. If the market behaved exactly as in the Black-Scholes theory, then $\sigma_{\text{imp}}(K) = \sigma$ for all values of $K$, hence the graph of the function $K \to \sigma_{\text{imp}}(K)$ would be a straight horizontal line. Given that real markets do not satisfy exactly the assumptions in the Black-Scholes theory, what can we say about the graph of the function $K \to \sigma_{\text{imp}}(K)$? Remarkably, it has been found that there exists recurrent convex shapes for the graph of this function, which are known as **volatility smile** and **volatility skew**, see Figures 6.2-6.3. The minimum of the volatility smile is reached at the strike price $K \approx S(t)$, i.e., when the call is at the money. This behavior indicates that the more the call is far from being at the money, the more it will be overpriced. Volatility smiles and skews have been found in the market especially after the crash in 1987 (Black Monday), indicating that this event led investors to be more cautious when trading on options that are in or out of the money. Devise mathematical models of stochastic volatility and asset prices able to reproduce volatility curves is an active research topic in mathematical finance.

**Exercise 6.17** (?). *What can we infer about the investors behavior from the volatility skew?*

## 6.6 Standard European derivatives on a dividend-paying stock

In this section we consider Black-Scholes markets with a dividend-paying stock. This means that at some time $t_0 \in (0, T)$ the price of the stock decreases of a fraction $a \in (0, 1)$ of its price

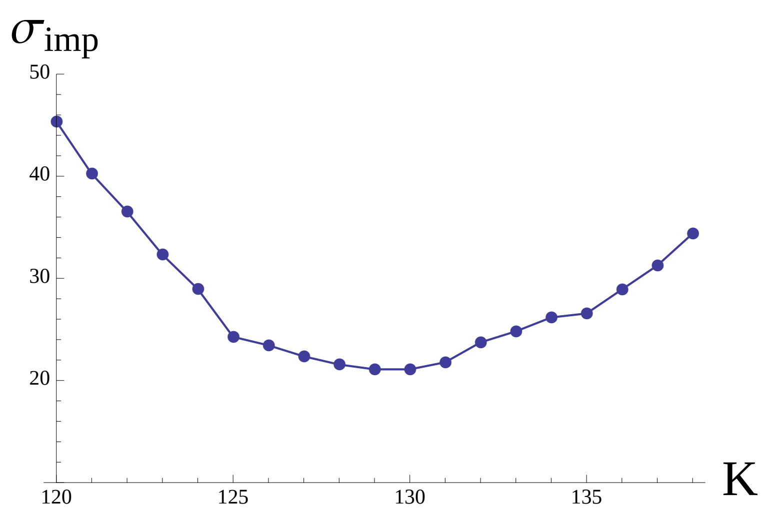



Figure 6.2: Volatility Smile of a call option on Apple expiring May $15^{th}$, 2015. The data were taken on May $12^{th}$, when the Apple stock quoted 126.34 dollars.

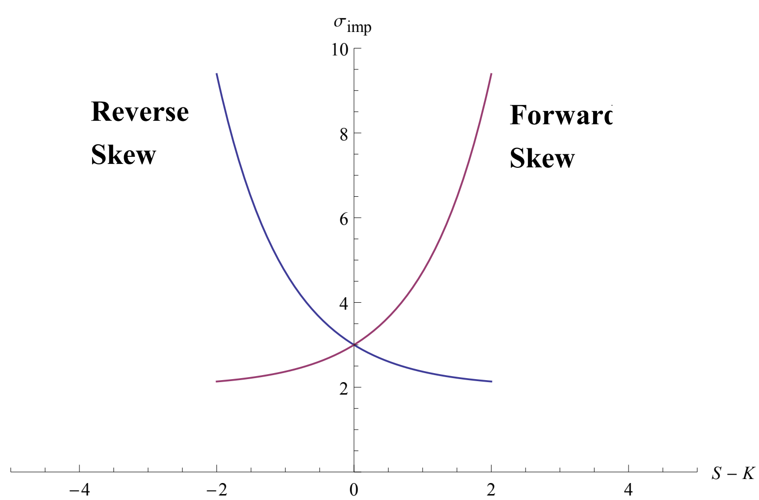



Figure 6.3: Volatility skews (not from real data!)

immediately before $t_0$, the difference being deposited in the account of the shareholders[1]. Letting $S(t_0^-) = \lim_{t \to t_0^-} S(t)$, we then have

$$S(t_0) = S(t_0^-) - aS(t_0^-) = (1-a)S(t_0^-). \tag{6.36}$$

[1] The dividend is expressed in percentage of the price of the stock. For instance, $a = 0.03$ means that the dividend paid is 3%.

We assume that on each of the intervals $[0,t_0)$, $[t_0,T]$, the stock price follows a geometric Brownian motion, namely,

$$S(s) = S(t)e^{\alpha(s-t)+\sigma(W(s)-W(t))}, \quad t \in [0,t_0),\ s \in [t,t_0) \tag{6.37}$$

$$S(s) = S(u)e^{\alpha(s-u)+\sigma(W(s)-W(u))}, \quad u \in [t_0,T],\ s \in [u,T]. \tag{6.38}$$

**Theorem 6.9** (∗)**.** *Consider the standard European derivative with pay-off* $Y = g(S(T))$ *and maturity* $T$*. Let* $\Pi_Y^{(a,t_0)}(t)$ *be the Black-Scholes price of the derivative at time* $t \in [0,T]$ *assuming that the underlying stock pays the dividend* $aS(t_0^-)$ *at time* $t_0 \in (0,T)$*. Then*

$$\Pi_Y^{(a,t_0)}(t) = \begin{cases} v(t,(1-a)S(t)), & \text{for } t < t_0, \\ v(t,S(t)), & \text{for } t \geq t_0, \end{cases}$$

*where* $v(t,x)$ *is the pricing function in the absence of dividends, which is given by* (6.19).

*Proof.* Using $\frac{S(T)}{S(t)} = e^{\alpha\tau+\sigma(W(T)-W(t))}$, we can rewrite (6.20) in the form

$$\Pi_Y(t) = e^{-r\tau}\mathbb{E}[g(S(T)e^{(r-\frac{\sigma^2}{2}-\alpha)\tau})]. \tag{6.39}$$

We want to express $S(T)$ in this formula in terms of $S(t)$ in the cases $0 \leq t < t_0$ and $t_0 \leq t \leq T$. Taking the limit $s \to t_0^-$ in (6.37) and using the continuity of the paths of the Brownian motion we find

$$S(t_0^-) = S(t)e^{\alpha(t_0-t)+\sigma(W(t_0)-W(t))}, \quad t \in [0,t_0).$$

Replacing in (6.36) we obtain

$$S(t_0) = (1-a)S(t)e^{\alpha(t_0-t)+\sigma(W(t_0)-W(t))}, \quad t \in [0,t_0).$$

Hence, letting $(s,u) = (T,t_0)$ and $(s,u) = (T,t)$ into (6.38), we find

$$S(T) = \begin{cases} (1-a)S(t)e^{\alpha\tau+\sigma(W(T)-W(t))} & \text{for } t \in [0,t_0), \\ S(t)e^{\alpha\tau+\sigma(W(T)-W(t))} & \text{for } t \in [t_0,T]. \end{cases} \tag{6.40}$$

By the definition of Black-Scholes price in the form (6.39) and denoting $G = (W(T) - W(t))/\sqrt{\tau}$, we obtain

$$\Pi_Y^{(a,t_0)}(t) = e^{-r\tau}\mathbb{E}[g((1-a)S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}], \quad \text{for } t \in [0,t_0),$$

$$\Pi_Y^{(a,t_0)}(t) = e^{-r\tau}\mathbb{E}[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}G}], \quad \text{for } t \in [t_0,T].$$

As $G \in \mathcal{N}(0,1)$, the result follows. □

We conclude that for $t \geq t_0$, i.e., after the dividend has been paid, the Black-Scholes price function of the derivative is again given by (6.19), while for $t < t_0$ it is obtained by replacing $x$ with $(1-a)x$ in (6.19). To see the effect of this change, suppose that the derivative is a call option; let $C(t,x)$ be the Black-Scholes price function in the absence of dividends and $C_a(t,x)$ be the price function in the case that a dividend is paid at time $t_0$. Then, according to Theorem 6.9,

$$C_a(t,x) = \begin{cases} C(t,(1-a)x), & \text{for } t < t_0, \\ C(t,x), & \text{for } t \geq t_0. \end{cases}$$

Since $\partial_x C > 0$ (see Theorem 6.7), it follows that $C_a(t,x) < C(t,x)$, for $t < t_0$, that is to say, the payment of a dividend makes the call option on the stock less valuable (i.e., cheaper) than in the absence of dividends until the dividend is paid.

**Exercise 6.18** (?). *Give an intuitive explanation for the property just proved for call options on a dividend paying stock.*

**Exercise 6.19** (•). *A standard European derivative pays the amount $Y = (S(T) - S(0))_+$ at time of maturity $T$. Find the Black-Scholes price $\Pi_Y(0)$ of this derivative at time $t = 0$ assuming that the underlying stock pays the dividend $(1-e^{-rT})S(\frac{T}{2}-)$ at time $t = \frac{T}{2}$. Compute the probability of positive return for a constant portfolio which is short 1 share of the derivative and short $S(0)e^{-rT}$ shares of the risk-free asset (assume $B(0) = 1$).*

**Exercise 6.20.** *Derive the Black-Scholes price of the derivative with pay-off $Y = g(S(T))$, assuming that the underlying pays a dividend at each time $t_1 < t_2 < \cdots < t_M \in [0,T]$. Denote by $a_i$ the dividend paid at time $t_i$, $i = 1,\dots,M$.*

## 6.7 Optimal exercise time of American calls on dividend-paying stocks

The Black-Scholes pricing theory for American derivatives is quite complicated and beyond the purpose of these notes. In this section we prove a property of American calls on a dividend-paying stock, which depends only on the following facts:

(i) An American derivative is at least as valuable as its European counterpart and its intrinsic value;

(ii) If it is not optimal to exercise the American derivative for $t \in [t_0, T]$, where $t_0 \in [0,T]$, then the fair value of the American derivative and of its European counterpart are the same for $t \in [t_0, T]$.

These facts can be proved rigorously and hold regardless of whether the underlying stock pays or not a dividend. They are of course quite intuitive properties (see Chapter 1) and will be used without further comment in this section. We also assume throughout this section that the interest rate $r$ of the money market is non-negative.

Let $\widehat{C}_a(t, S(t), K, T)$ denote the Black-Scholes price at time $t$ of the American call with strike $K$ and maturity $T$ assuming that the underlying stock pays the dividend $aS(t_0^-)$ at time $t_0 \in$

$(0, T)$. We denote by $C_a(t, S(t), K, T)$ the Black-Scholes price of the corresponding European call. We omit the subscript $a$ to denote prices in the absence of dividends. Moreover replacing the letter $C$ with the letter $P$ gives the price of the corresponding put option. We say that it is optimal to exercise the American call at time $t$ if its Black-Scholes price at this time equals the intrinsic value of the call, i.e., $\widehat{C}_a(t, S(t), K, T) = (S(t) - K)_+$.

**Theorem 6.10** (∗). *Consider the American call with strike $K$ and expiration date $T$ and assume that the underlying stock pays the dividend $aS(t_0^-)$ at the time $t_0 \in (0, T)$. Then*

$$\widehat{C}_a(t, S(t), K, T) > (S(t) - K)_+, \quad \textit{for } t \in [t_0, T),$$

*i.e., it is not optimal to exercise the American call prior to maturity after the dividend is paid. Moreover, there exists $\delta > 0$ such that, if*

$$S(t_0^-) > \max(\frac{\delta}{1 - a}, K),$$

*then the equality*

$$\widehat{C}_a(t_0^-, S(t_0^-), K, T) = (S(t_0^-) - K)_+$$

*holds, and so it is optimal to exercise the American call "just before" the dividend is to be paid.*

*Proof.* For the first claim we can assume $(S(t) - K)_+ = S(t) - K$, otherwise the American call is out of the money and so it is clearly not optimal to exercise. By Theorem 6.9 we have

$$C_a(t, S(t), K, T) = C(t, S(t), K, T), \quad P_a(t, S(t), K, T) = P(t, S(t), K, T), \quad \text{for } t \geq t_0.$$

Hence, by Theorem 6.5, the put-call parity holds after the dividend is paid:

$$C_a(t, S(t), K, T) = P_a(t, S(t), K, T) + S(t) - Ke^{-r(T-t)}, \quad t \geq t_0.$$

Thus, for $t \in [t_0, T)$,

$$\widehat{C}_a(t, S(t), K, T) \geq C_a(t, S(t), K, T) > S(t) - K = (S(t) - K)_+,$$

where we used that $P(t, S(t), K, T) > 0$ and $r \geq 0$. This proves the first part of the theorem, i.e., the fact that it is not optimal to exercise the American call prior to expiration after the dividend has been paid. In particular

$$\widehat{C}_a(t, S(t), K, T) = C_a(t, S(t), K, T), \quad \text{for } t \geq t_0. \tag{6.41}$$

Next we show that it is optimal to exercise the American call "just before" the dividend is paid, i.e., $\widehat{C}_a(t_0^-, S(t_0^-), K, T) = (S(t_0^-) - K)_+$, provided the price of the stock is sufficiently high. Of course it must be $S(t_0^-) > K$. Assume first that $\widehat{C}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K$; then, owing to (6.41), $\widehat{C}_a(t_0^-, S(t_0^-), K, T) = C_a(t_0^-, S(t_0^-), K, T)$ (buying the American call just before the dividend is paid is not better than buying the European call, since it is never optimal to exercise the derivative prior to expiration). By Theorem (6.9) we have $C_a(t_0^-, S(t_0^-), K, T) = C(t_0^-, (1-a)S(t_0^-), K, T) = C(t_0, (1-a)S(t_0^-), K, T)$, where for the latter equality we used the continuity in time of the Black-Scholes price function in the absence of dividends. Since $(1-a)S(t_0^-) = S(t_0)$, then

$$\widehat{C}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K \Rightarrow \widehat{C}_a(t_0^-, S(t_0^-), K, T) = C(t_0, S(t_0), K, T).$$

Hence

$$\widehat{C}_a(t_0^-, S(t_0^-), K, T) > S(t_0^-) - K \Rightarrow C(t_0, S(t_0), K, T) > S(t_0^-) - K = S(t_0) + (1-a)S(t_0^-) - K.$$

Therefore, taking the contrapositive statement,

$$C(t_0, S(t_0), K, T) \leq S(t_0) + (1-a)S(t_0^-) - K \Rightarrow \widehat{C}_a(t_0^-, S(t_0^-), K, T) = S(t_0^-) - K. \quad (6.42)$$

Next we remark that the function $x \to C(t, x, K, T) - x$ is decreasing (since $\Delta = \partial_x C = \Phi(d_1) < 1$, see Theorem 6.7), and

$$\lim_{x \to 0^+} C(t, x, K, T) - x = 0,$$

$$\lim_{x \to +\infty} C(t, x, K, T) - x = \lim_{x \to +\infty} P(t, x, K, T) - Ke^{-r(T-t)} = -Ke^{-r(T-t)},$$

see Exercise 6.9. Thus if $(1-a)S(t_0^-) - K > -Ke^{-r(T-t)}$, i.e., $S(t_0^-) > (1-a)^{-1}K(1 - e^{-r(T-t)}$, there exists $\omega$ such that if $S(t_0) > \omega$, i.e., $S(t_0^-) > \omega/(1-a)$, then the inequality $C(t_0, S(t_0), K, T) \leq S(t_0) + (1-a)S(t_0^-) - K$ holds. It follows by (6.42) that for such values of $S(t_0^-)$ it is optimal to exercise the call "at time $t_0^-$". Letting $\delta = \max(\omega, K(1 - e^{-r(T-t)})$ concludes the proof of the theorem. □

**Exercise 6.21.** *Prove that it is not optimal to exercise the American call at time $t \in [0, t_0)$ if $S(t) < \frac{K}{a}(1 - e^{-r(T-t)})$.*

# Appendix A

# The Markowitz portfolio theory

Consider a constant portfolio position in some time interval $[0, T]$. We assume that the portfolio is invested in $n$ risky assets $\mathcal{U}_1, \dots, \mathcal{U}_n$ with prices $\Pi^{\mathcal{U}_i}(t)$, $i = 1, \dots, N$ and a risk-free asset $\mathcal{U}_{n+1}$ with value $B(t) = B_0 e^{rt}$, where $r > 0$ is the instantaneous interest rate. For notational convenience we let $\Pi^{\mathcal{U}_{n+1}}(t) = B(t)$, so that $\Pi^{\mathcal{U}_i}(t)$, $i = 1, \dots, n+1$ now denotes the price of every asset in the portfolio. The relative return of the asset $i$ is defined by

$$R_i = \frac{\Pi^{\mathcal{U}_i}(T) - \Pi^{\mathcal{U}_i}(0)}{\Pi^{\mathcal{U}_i}(0)}, \quad i = 1, \dots, n+1.$$

For $i = 1, \dots, n$, $R_i$ is a random variable, while for $i = n+1$ it is the deterministic constant:

$$R_{n+1} = \frac{B_0 e^{rT} - B_0}{B_0} = e^{rT-1} := \rho > 0.$$

As usual, the price of the assets at time $t = 0$ are supposed to be known.

The problem under study in this chapter is the following: Given an investor with initial capital $K > 0$, what is the best way to distribute this wealth among the $n+1$ assets in order to maximize the expected return and, at the same time, minimize the expected risk? To solve this problem we need first to introduce some notation. Let $a_i \in \mathbb{R}$ denote the number of shares of the asset $\mathcal{U}_i$ in the portfolio. The initial value of the portfolio is

$$V(0) = \sum_{i=1}^{n+1} a_i \Pi^{\mathcal{U}_i}(0) = K.$$

Letting

$$\pi_i = \frac{a_i \Pi^{\mathcal{U}_i}(0)}{K}, \quad i = 1, \dots, n+1, \tag{A.1}$$

we get

$$\sum_{i=1}^{n+1} \pi_i = 1. \tag{A.2}$$

The value of the portfolio at time $t = T$ is

$$V(T) = \sum_{i=1}^{n+1} a_i \Pi^{\mathcal{U}_i}(T) = \sum_{i=1}^{n+1} a_i \Pi^{\mathcal{U}_i}(0)(1 + R_i).$$

The relative return of the portfolio in the interval $[0, T]$ is given by

$$R = \frac{V(T) - V(0)}{V(0)} = \frac{V(T)}{K} - 1.$$

After simple calculations we obtain

$$R = \sum_{i=1}^{n+1} \pi_i R_i, \tag{A.3}$$

where we used (A.1)-(A.2). The relative return of the portfolio is a random variable and taking its expectation we obtain

$$\mathbb{E}[R] = \sum_{i=1}^{n+1} \pi_i \mathbb{E}[R_i] = \sum_{i=1}^{n} \pi_i \mathbb{E}[R_i] + \pi_{n+1} \mathbb{E}[R_{n+1}].$$

Using (A.2) and that $\mathbb{E}[R_{n+1}] = R_{n+1} = \rho$ we obtain

$$\mathbb{E}[R] = \sum_{i=1}^{n} \pi_i \mu_i + \left(1 - \sum_{i=1}^{n} \pi_i\right) \rho, \tag{A.4}$$

where we set

$$\mu_i = \mathbb{E}[R_i], \quad \text{for } i = 1, \dots, n.$$

Hence the portfolio should be chosen to maximize (A.4) and at the same time to minimize the portfolio risk. We measure the latter with the variance $\mathrm{Var}[R]$ of the relative return. To compute $\mathrm{Var}[R]$ we first observe that

$$R - \mathbb{E}[R] = \sum_{i=1}^{n+1} \pi_i R_i - \sum_{i=1}^{n+1} \pi_i \mathbb{E}[R_i] = \sum_{i=1}^{n} \pi_i (R_i - \mathbb{E}[R_i]),$$

hence

$$\begin{aligned}(R - \mathbb{E}[R])^2 &= \left(\sum_{i=1}^{n} \pi_i (R_i - \mathbb{E}[R_i])\right)^2 = \left(\sum_{i=1}^{n} \pi_i (R_i - \mathbb{E}[R_i])\right)\left(\sum_{j=1}^{n} \pi_j (R_j - \mathbb{E}[R_j])\right) \\ &= \sum_{i,j=1}^{n} \pi_i \pi_j (R_i - \mu_i)(R_j - \mu_j).\end{aligned}$$

Letting $C = (c_{ij})_{i,j=1\ldots n}$, $c_{ij} = \mathrm{Cov}(R_i, R_j)$, be the covariance matrix of the assets returns and $\pi = (\pi_1\ \pi_2 \ldots \pi_n)^T$, we obtain

$$\mathrm{Var}[R] = \mathbb{E}[(R - \mathbb{E}[R])^2] = \sum_{i,j=1}^{n} \pi_i \pi_j c_{ij} = \pi^T C \pi. \tag{A.5}$$

Our purpose is then to find a portfolio which maximizes (A.4) and minimizes (A.5). To this end we shall need the following simple result.

**Theorem A.1.** *Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ be random variables and $c_{ij} = \mathrm{Cov}(X_i, X_j)$. The covariance matrix $C = (c_{ij})_{i,j=1,\ldots,n}$ is symmetric and positive-semidefinite, i.e., $C = C^T$ and $\xi^T C \xi \geq 0$, for all $\xi \in \mathbb{R}^n$.*

**Exercise A.1.** *Prove the theorem.*

**Definition A.1.** *Given $\theta > 0$, let*

$$g_\theta(\pi_1, \ldots, \pi_n) = \mathbb{E}[R] - \theta \mathrm{Var}[R]. \tag{A.6}$$

*A portfolio $(a_1, \ldots, a_n, a_{n+1})$ is called a* **Markowitz portfolio** *if the function $g_\theta$ has a maximum at $\widehat{\pi} = (\widehat{\pi}_1, \ldots, \widehat{\pi}_n)$, where*

$$\widehat{\pi}_1 = \frac{a_1 S_1(0)}{K}, \ldots, \widehat{\pi}_n = \frac{a_n S_n(0)}{K}.$$

The parameter $\theta > 0$ is called the **risk aversion** of the investor. We shall see later that the higher is $\theta$ the more portfolio value is invested in the risk-free asset. Hence a large $\theta$ corresponds to a cautious investor, who favors "small risks" to "higher returns".

**Theorem A.2.** *Assume that the covariance matrix of the assets returns is positive-definite, i.e., $\xi^T C \xi > 0$, for all $\xi \neq 0$. Let $\Omega$ be the vector*

$$\Omega = \frac{1}{2\theta}(\mu_1 - \rho \quad \mu_2 - \rho \ \ \ldots \ \ \mu_n - \rho)^T.$$

*Then the function (A.6) has a unique maximum, which is attained at $\widehat{\pi} = C^{-1}\Omega$.*

*Proof.* Using (A.4) and (A.5) the function $g_\theta$ is

$$g_\theta(\pi) = \sum_{i=1}^{n} \mu_i \pi_i + \left(1 - \sum_{i=1}^{n} \pi_i\right) \rho - \theta \sum_{i,j=1}^{n} \pi_i \pi_j c_{ij}.$$

Hence

$$\frac{\partial g_\theta}{\partial \pi_k} = \mu_k - \rho - 2\theta \sum_{j=1}^{n} c_{kj} \pi_j.$$

Thus the stationary points, i.e., the solutions $\widehat{\pi}$ of $\nabla_\pi g_\theta(\widehat{\pi}) = 0$, satisfy

$$\sum_{i=1}^{n} c_{kj}\widehat{\pi}_j = \frac{1}{2\theta}(\mu_k - \rho), \quad \text{i.e.,} \quad C\widehat{\pi} = \Omega.$$

As $C$ is assumed to positive definite, then it is invertible and thus $C\widehat{\pi} = \Omega$ has the unique solution $\widehat{\pi} = C^{-1}\Omega$. To show that this stationary point is a maximum of $g_\theta$ we compute

$$\frac{\partial^2 g}{\partial \pi_k \partial \pi_l} = -2\theta c_{kl}.$$

Since $C$ is positive definite, then the Hessian $\nabla_\pi^2 g_\theta$ is negative definite and therefore the point $\widehat{\pi}$ is a maximum. This concludes the proof of the theorem. □

**Remark A.1.** Note that $\widehat{\pi}_i \to 0$, and so also $a_i \to 0$, as $\theta \to +\infty$, for all $i = 1, \dots, n$. Hence in this limit the entire capital $K$ is invested in the risk-free asset: $\pi_{n+1} \to 1$, as $\theta \to +\infty$, which is equivalent to $a_{n+1} \to K/\Pi^{\mathcal{U}_{n+1}}(0) = KB_0$, as $\theta \to +\infty$. Conversely, the smaller is $\theta$, the larger is the exposure of the investor on the risky assets.

As a special case, assume that $\Pi^{\mathcal{U}_1(T)}, \dots, \Pi^{\mathcal{U}_n(T)}$ are independent. Then $R_1(T), \dots, R_n(T)$ are also independent and therefore $\mathrm{Cov}(R_i, R_j) = 0$, for $i \neq j$. Hence

$$C = \mathrm{diag}(\mathrm{Var}[R_1], \mathrm{Var}[R_2], \dots, \mathrm{Var}[R_n]),$$

where $\mathrm{diag}(z_1, \dots, z_n)$ is the diagonal $n \times n$ matrix $A$ with diagonal elements $a_{ii} = z_i$, $i = 1, \dots n$. Therefore in this case the maximum of $g_\theta$ is attained at

$$\widehat{\pi}_i = \frac{\mathbb{E}[R_i] - \rho}{2\theta \mathrm{Var}[R_i]}. \tag{A.7}$$

## Example

Suppose that the risky assets are stocks and that the prices $\Pi^{\mathcal{U}_i}(t) = S_i(t)$ are independent and that each of them follows a binomial model:

$$S_i(j) = \begin{cases} S_i(j-1)e^{u_i}, & \text{with probability } p_i \\ S_i(j-1)e^{d_i}, & \text{with probability } = 1 - p_i \end{cases}, \quad j \in \mathcal{I} = \{1, \dots, N\}, \quad i = 1, \dots n.$$

Let us compute the Markowitz portfolio in the $N$ days period. We have

$$\mathbb{E}[R_i] = \mathbb{E}[S_i(0)^{-1}(S_i(N) - S_i(0))] = \frac{1}{S_i(0)}\mathbb{E}[S_i(N)] - 1.$$

By Theorem **??** we have $\mathbb{E}[S_i(N)] = S_i(0)(e^{d_i}p_i + e^{u_i}(1-p_i))^N$. Hence

$$\mathbb{E}[R_i] = (e^{d_i}p_i + e^{u_i}(1-p_i))^N - 1. \tag{A.8}$$

Moreover

$$\mathbb{E}[S_i(N)^2] = S_i(0)^2(e^{2d_i}p_i + e^{2u_i}(1-p_i))^N,$$

since $S_i(t)$ follows a binomial model with parameters $2u, 2d$. Hence

$$\mathrm{Var}[S_i(N)] = \mathbb{E}[S_i(N)^2] - \mathbb{E}[S_i(N)]^2 = S_i(0)^2[(e^{2d_i}p_i + e^{2u_i}(1-p_i))^N - (e^{d_i}p_i + e^{u_i}(1-p_i))^{2N}]$$

and therefore

$$\mathrm{Var}[R_i] = \frac{1}{S_i(0)^2}\mathrm{Var}[S_i(N)] = (e^{2d_i}p_i + e^{2u_i}(1-p_i))^N - (e^{d_i}p_i + e^{u_i}(1-p_i))^{2N}. \qquad \text{(A.9)}$$

Replacing (A.8) and (A.9) into (A.7), and then inverting (A.1), we obtain the desired Markowitz portfolio $(a_1, \dots, a_{n+1})$.

**Exercise A.2** (•). *Consider a 2-period binomial model with the following parameters*

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad p = \frac{1}{2}.$$

*Assume further that $S(0) = 36$, and that the interest rate of the money market is zero. Consider also the European derivative with pay-off*

$$Y = (S(2) - 28)_+ - 2(S(2) - 32)_+ + (S(2) - 36)_+$$

*and time of maturity $T = 2$. Compute the fair value of the derivative at $t = 0$. Assume now that an investor with risk aversion $\theta = \frac{1}{36}$ wants to distribute the initial wealth $K = 1000$ in the following assets: the stock, the derivative, and a risk-free asset with interest $r$ such that $e^{2r} = 10/9$. Derive the corresponding Markowitz portfolio.*

**Exercise A.3** ($\star$). *Consider two stocks with prices*

$$S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_1 W_1(t)}, \quad S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_2 W_2(t)}$$

*where $\sigma_1, \sigma_2 > 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and $W_1(t), W_2(t)$ are two Brownian motions. Let $T > 0$ and assume that $W_1(T)$ and $W_2(T)$ are independent random variables. Compute the Markowitz portfolio of an investor with initial capital $K > 0$ and risk aversion $\theta$ who wants to invest in the stocks and in a money market with interest $r > 0$ during the interval of time $[0, T]$.*

# Appendix B

# Solutions to selected exercises

## Exercise 1.7

To solve this exercise it is useful to begin by drawing the pay-off as a function of $S(T)$. Since

$$Y = \min\left[(S(T) - K_1)_+, (K_2 - S(T))_+\right],$$

where $K_2 > K_1$ and $(x)_+ = \max(0, x)$, then we first draw the functions $S(T) \to (S(T)-K_1)_+$ and $S(T) \to (K_2 - S(T))_+$ and then we take their minimum.

Now let $\mathcal{A}$ be a portfolio that consists of one share of the derivative and let $V_{\mathcal{A}}(t)$ be its value at time $t \in [0, T]$. The exercise asks to derive a portfolio $\mathcal{B}$ consisting of European calls and puts which replicates the value of $\mathcal{A}$, i.e., such that $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$, for all $t \in [0, T]$. By the dominance principle (in particular by (b) of 1.1) it is enough to find the portfolio $\mathcal{B}$ in such a way that $V_{\mathcal{A}}(T) = V_{\mathcal{B}}(T)$. Clearly $V_{\mathcal{A}}(T) = Y$, since the value of a derivative at the expiration date always equals the pay-off. So we we have to find a combination of pay-off functions of puts and calls such that their sum equal $Y$. By using the graph of the pay-off function it is easy to see that

$$Y = (S(T) - K_1)_+ - 2(S(T) - \frac{K_1 + K_2}{2})_+ + (S(T) - K_2)_+$$

Even without making any drawing, the previous identity can be verified by direct calculation. Hence, letting

$$\begin{aligned}
\mathcal{U}_1 &= \text{European call with strike } K_1 \\
\mathcal{U}_2 &= \text{European call with strike } (K_1 + K_2)/2 \\
\mathcal{U}_3 &= \text{European call with strike } K_2
\end{aligned}$$

where for all calls the expiration date is $T$, we take

$$\mathcal{B} = (1, -2, 1).$$

This concludes the solution of the exercise.

## Exercise 1.9

Consider the following assets:

- $\mathcal{U}_1 \equiv$ Contract
- $\mathcal{U}_2 \equiv$ Risk-free asset with initial value 1
- $\mathcal{U}_3 \equiv$ European Call with strike $K$ and maturity $T$

Consider the two portfolios

$$\mathcal{A} = (0, Ne^{-rT}, \alpha N), \quad \mathcal{B} = (1, 0, 0).$$

We want to show that the portfolio $\mathcal{A}$ on the call and the risk-free asset replicates the value of the contract. To this purpose we observe that

$$V_{\mathcal{A}}(t) = Ne^{-r(T-t)} + \alpha N C(t, S(t), K, T), \tag{B.1}$$

while $V_{\mathcal{B}}(t) = N$ by assumption. Letting $t = T$ in (B.1) we obtain

$$V_{\mathcal{A}}(T) = N + \alpha N (S(T) - K)_+ = V_{\mathcal{B}}(T),$$

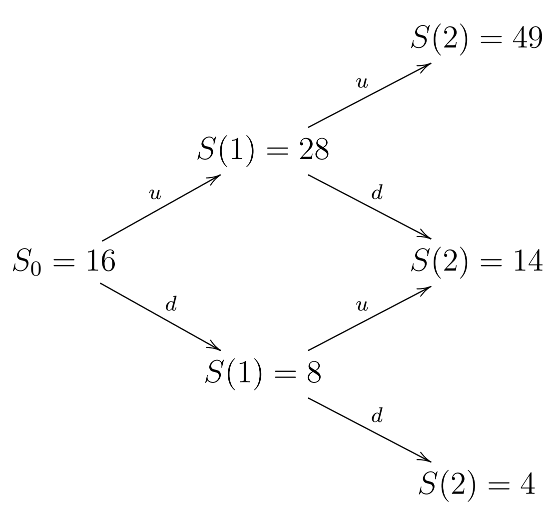
hence, by the dominance principle, $V_{\mathcal{A}}(t) = V_{\mathcal{B}}(t)$. Using (B.1) and $V_{\mathcal{B}}(t) = N$ we obtain

$$Ne^{-r(T-t)} + \alpha N C(t, S(t), K, T) = N$$

and since $C(t, S(t), K, T) > 0$ and $N \neq 0$, we obtain (1.11).

## Exercise 3.2

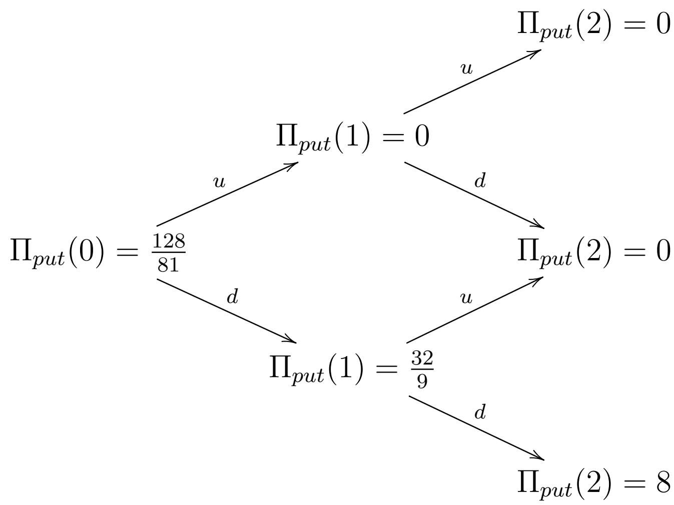
The binomial tree for the price of the stock is



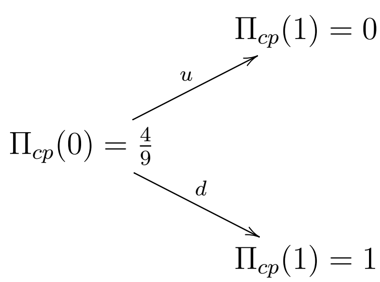
Moreover $q_u = \frac{e^r - e^d}{e^u - e^d} = 1/2 = q_d$. Using the recurrence formula $\Pi_Y(2) = Y$,

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)), \quad t = 0, 1,$$

we find that the price $\Pi_{put}(t)$ of the put option is




The pay-off for the call on the put at time $t = T_1 = 1$ is $(\Pi_{put} - K_1)_+$. Note that since $\Pi_{put}(1) = g(S(1))$, then the call on the put can be treated as a standard European derivative on the stock expiring at time $T_1 = 1$. Hence the price $\Pi_{cp}(t)$ of the call on the put is




This completes the first part of the exercise. Now, the return of a portfolio with +1 share of the call on the put is path dependent. We have

$$R(u,u) = 0 - \frac{4}{9} = -\frac{4}{9}, \quad R(u,d) = 0 - \frac{4}{9} = -\frac{4}{9}$$

$$R(d,u) = 0 - \frac{4}{9} - \frac{23}{9} = -3, \quad R(d,d) = 8 - \frac{4}{9} - \frac{23}{9} = 5.$$

Hence the probability of positive return for the buyer is is

$$\mathbb{P}(R > 0) = \mathbb{P}(S^{(u,u)}) = \left(\frac{3}{4}\right)^2 = 75\%.$$

## Exercise 3.3

We have the general formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x), \tag{B.2}$$

where $(q_u, q_d)$ is the risk neutral probability, $N_u(x)$ is the number of "$u$" in the path $x$ and $N_d(x) = N - N_u(x)$ is the number of "$d$" in the path $x$. Moreover $Y(x)$ denotes the pay-off as a function of the path of the stock price. The exercise tells us that

$$Y(x) = S(N, x), \quad \text{if } x = x_* = (u, u, \dots u), \text{ i.e., } x_i = u \text{ for all } i = 1, \dots N,$$

while $Y(x) = S(0)$ for $x \neq x_*$. Moreover, since $S(N, x_*) = S(0)e^{Nu}$, then

$$Y(x_*) = S(0)e^{Nu}.$$

Since in addition $N_u(x_*) = N$, we can rewrite the sum (B.2) as

$$\begin{aligned} \Pi_Y(0) &= e^{-rN}(q_u)^{N_u(x_*)}(q_d)^{N_d(x_*)}Y(x_*) + e^{-rN} \sum_{x \neq x_*} (q_u)^{N_u(x)}(q_d)^{N_d(x)}Y(x) \\ &= e^{-rN}(q_u)^N S(0)e^{Nu} + e^{-rN} \sum_{x \neq x_*} (q_u)^{N_u(x)}(q_d)^{N_d(x)}Y(x). \end{aligned} \tag{B.3}$$

Next we compute the sum on $x \neq x_*$. First replacing $N_d(x) = N - N_u(x)$ and $Y(x) = S(0)$ we have

$$\sum_{x \neq x_*} (q_u)^{N_u(x)}(q_d)^{N_d(x)}Y(x) = S(0)(q_d)^N \sum_{x \neq x_*} \left(\frac{q_u}{q_d}\right)^{N_u(x)}. \tag{B.4}$$

Now, $N_u(x)$ takes value in $\{0, 1, \dots, N-1\}$ in (B.4); it cannot be equal to $N$ because the only element in $\{u, d\}^N$ for which $N_u(x) = N$ is $x_*$, but this element is not taken into account in the sum that we are computing. Using that the number of $x \in \{u, d\}^N$ for which $N_u(x) = k$ is given by the binomial coefficient $\binom{N}{k}$, we obtain

$$\sum_{x \neq x_*} \left(\frac{q_u}{q_d}\right)^{N_u(x)} = \sum_{k=0}^{N-1} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k.$$

Adding and subtracting the term $k = N$ (where we use that $\binom{N}{N} = 1$), we find

$$\begin{aligned} \sum_{x \neq x_*} \left(\frac{q_u}{q_d}\right)^{N_u(x)} &= \sum_{k=0}^{N-1} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k \\ &= \sum_{k=0}^{N} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k - \left(\frac{q_u}{q_d}\right)^N \\ &= \left(1 + \frac{q_u}{q_d}\right)^N - \left(\frac{q_u}{q_d}\right)^N, \end{aligned}$$

where for the last equality we used the binomial theorem: $(1+a)^N = \sum_{k=0}^{N} \binom{N}{k} a^k$. Substituting into (B.4) and using that $q_u + q_d = 1$ we obtain

$$\sum_{x \neq x_*} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x) = S(0)\big(1 - (q_u)^N\big).$$

Finally, replacing in (B.3) we find

$$\Pi_Y(0) = e^{-rN} S(0) \left[ (q_u)^N e^{Nu} + 1 - (q_u)^N \right].$$

## Exercise 3.5

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} \log(S(N)/S(0)),$$

by Definition 3.3 (at $t = 0$). Replacing $S(N) = S(0)e^{N_u(x)u + N_d(x)d}$ we obtain

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} (N_u(x)u + N_d(x)d).$$

Replacing $N_d(x) = N - N_u(x)$ we obtain

$$\begin{aligned}
\Pi_Y(0) &= e^{-rN} (q_d)^N \sum_{x \in \{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)} (N_u(x)(u-d) + Nd) \\
&= e^{-rN} (q_d)^N (u-d) \sum_{x \in \{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)} N_u(x) + Nde^{-rN} (q_d)^N \sum_{x \in \{u,d\}^N} (\frac{q_u}{q_d})^{N_u(x)}
\end{aligned}$$

Letting $N_u(x) = k$ we obtain

$$\Pi_Y(0) = e^{-rN} (q_d)^N (u-d) \sum_{k=0}^{N} \binom{N}{k} k (\frac{q_u}{q_d})^k + Nde^{-rN} (q_d)^N \sum_{k=0}^{N} \binom{N}{k} (\frac{q_u}{q_d})^k \tag{B.5}$$

Using the binomial theorem, the second sum is

$$\sum_{k=0}^{N} \binom{N}{k} (\frac{q_u}{q_d})^k = (1 + \frac{q_u}{q_d})^N = \frac{1}{(q_d)^N}$$

In the first sum we use the identity

$$k\binom{N}{k} = N\binom{N-1}{k-1}.$$

Hence

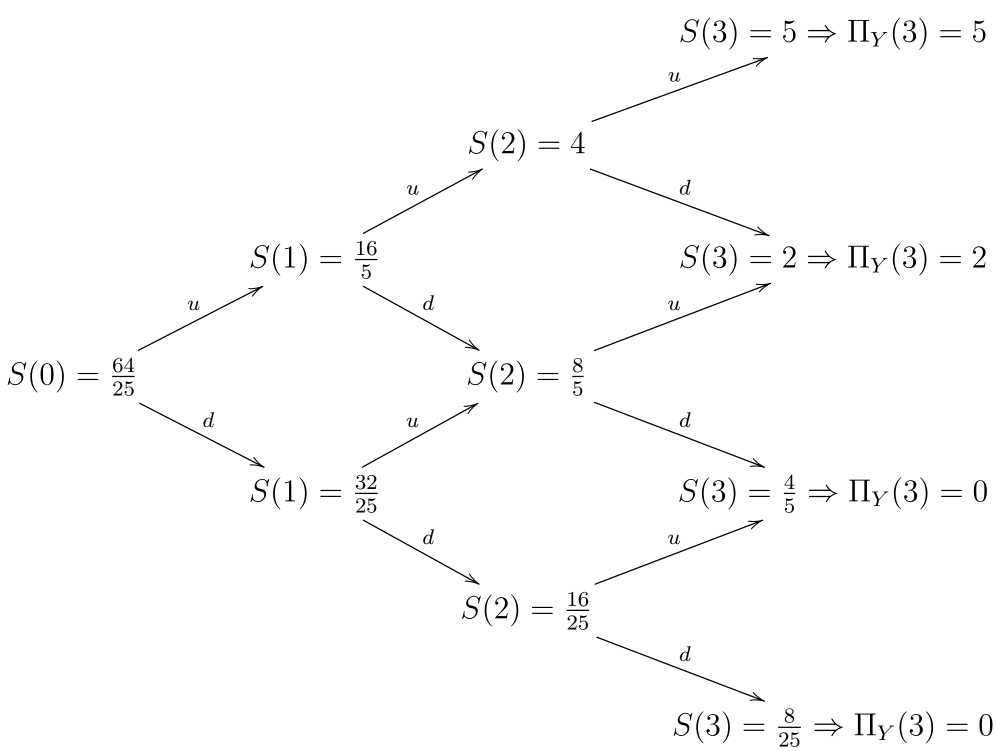
$$\begin{aligned}
\sum_{k=0}^{N}\binom{N}{k}k(\frac{q_u}{q_d})^k &= \sum_{k=1}^{N}\binom{N}{k}k(\frac{q_u}{q_d})^k \\
&= N\sum_{k=1}^{N}\binom{N-1}{k-1}(\frac{q_u}{q_d})^k = N\sum_{j=0}^{N-1}\binom{N-1}{j}(\frac{q_u}{q_d})^{j+1} \\
&= N\frac{q_u}{q_d}\sum_{j=0}^{N-1}\binom{N-1}{j}(\frac{q_u}{q_d})^j = N\frac{q_u}{q_d}(1+\frac{q_u}{q_d})^{N-1} \\
&= N\frac{q_u}{(q_d)^N}
\end{aligned}$$

Replacing in (B.5) we find

$$\begin{aligned}
\Pi_Y(0) &= e^{-rN}(q_d)^N(u-d)N\frac{q_u}{(q_d)^N} + Nde^{-rN}(q_d)^N\frac{1}{(q_d)^N} \\
&= Ne^{-rN}(q_u(u-d)+d) = Ne^{-rN}(q_u u + d(1-q_u)) = Ne^{-rN}(q_u u + q_d d).
\end{aligned}$$

## Exercise 3.7

We start by writing down the diagram of the stock price and the value of the derivative at time of maturity $T = 3$ (which is equal to the pay-off)



The parameters of the binomial model are such that

$$q_u = \frac{2}{3}, \quad q_d = \frac{1}{3}, \quad r = 0.$$

To compute the price of the derivative at the times $t \in \{0, 1, 2\}$ we use the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)) = \frac{2}{3}\Pi_Y^u(t+1) + \frac{1}{3}\Pi_Y^d(t+1), \quad t \in \{0, 1, 2\}.$$

Hence at time $t = 2$ we have

$$S(2) = 4 \Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 2 = 4$$
$$S(2) = \frac{8}{5} \Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 0 = \frac{4}{3}$$
$$S(2) = \frac{16}{25} \Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 0 = 0.$$

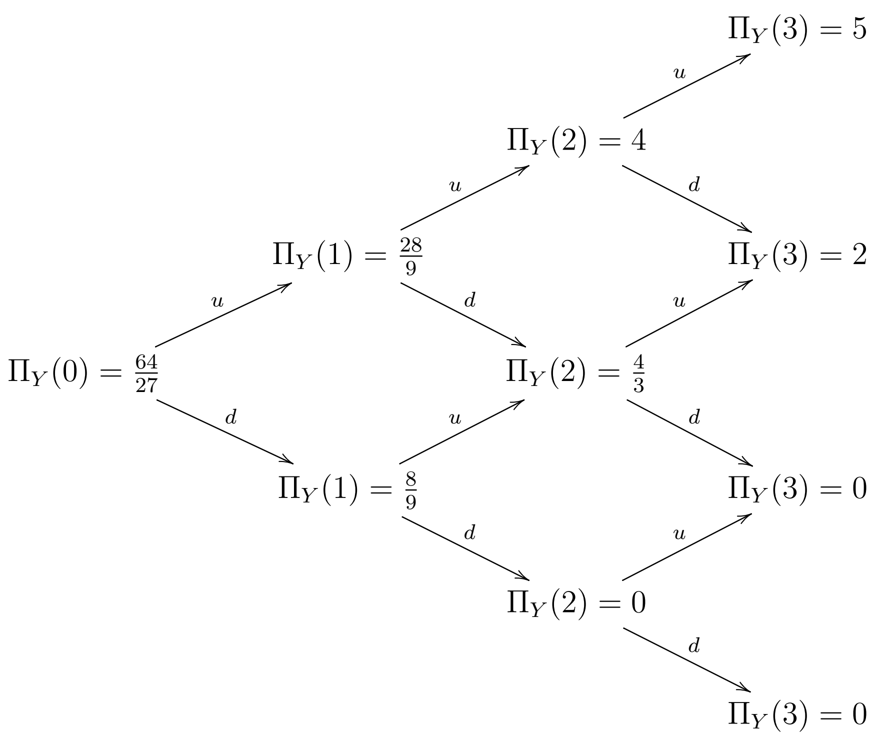
At time $t = 1$ we have

$$S(1) = \frac{16}{5} \Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot \frac{4}{3} = \frac{28}{9}$$
$$S(1) = \frac{32}{25} \Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot 0 = \frac{8}{9}$$

and at time $t = 0$ we have

$$\Pi_Y(0) = \frac{2}{3} \cdot \frac{28}{9} + \frac{1}{3} \cdot \frac{8}{9} = \frac{64}{27}$$

Hence we obtain the following diagram for the derivative price



This concludes the first part of the exercise. To compute the number of shares of the underlying asset in the hedging portfolio we use the formula

$$h_S(t+1) = \frac{1}{S(t)} \frac{\Pi_Y^u(t+1) - \Pi_Y^d(t+1)}{e^u - e^d}$$

for $t = 1, 2$ and $h_S(0) = h_S(1)$, where we recall that $h_S(t+1)$ is the position in the interval $(t, t+1]$. Letting $t = 2$ we obtain

$$h_S(3) = \frac{4}{3} \frac{\Pi_Y^u(3) - \Pi_Y^d(3)}{S(2)},$$

whence

$$S(2) = 4 \Rightarrow h_S(3) = \frac{4}{3} \cdot \frac{5-2}{4} = 1$$
$$S(2) = \frac{8}{5} \Rightarrow h_S(3) = \frac{4}{3} \cdot \frac{2-0}{8/5} = \frac{5}{3}$$
$$S(2) = \frac{16}{25} \Rightarrow h_S(3) = 0.$$

Likewise

$$h_S(2) = \frac{4}{3} \frac{\Pi_Y^u(2) - \Pi_Y^d(2)}{S(1)}.$$

Hence

$$S(1) = \frac{16}{5} \Rightarrow h_S(2) = \frac{4}{3} \cdot \frac{4 - 4/3}{16/5} = \frac{10}{9}$$
$$S(1) = \frac{32}{25} \Rightarrow h_S(2) = \frac{4}{3} \cdot \frac{4/3 - 0}{32/25} = \frac{25}{18}$$

and finally

$$h_S(0) = h_S(1) = \frac{4}{3} \frac{\Pi_Y^u(1) - \Pi_Y^d(1)}{S(0)} = \frac{4}{3} \cdot \frac{28/9 - 8/9}{64/25} = \frac{125}{108}.$$

This concludes the second part of the exercise. Consider now a constant portfolio with -1 shares of the derivative. The return of this portfolio is positive if the value of the derivative at the expiration date is smaller than the initial value. This happens along all paths except $x = (u, u, u)$, hence the probability that the return of this portfolio be positive is $1 - p^3 = 1 - 1/8 = 7/8 = 87.5\%$.

## Exercise 4.4

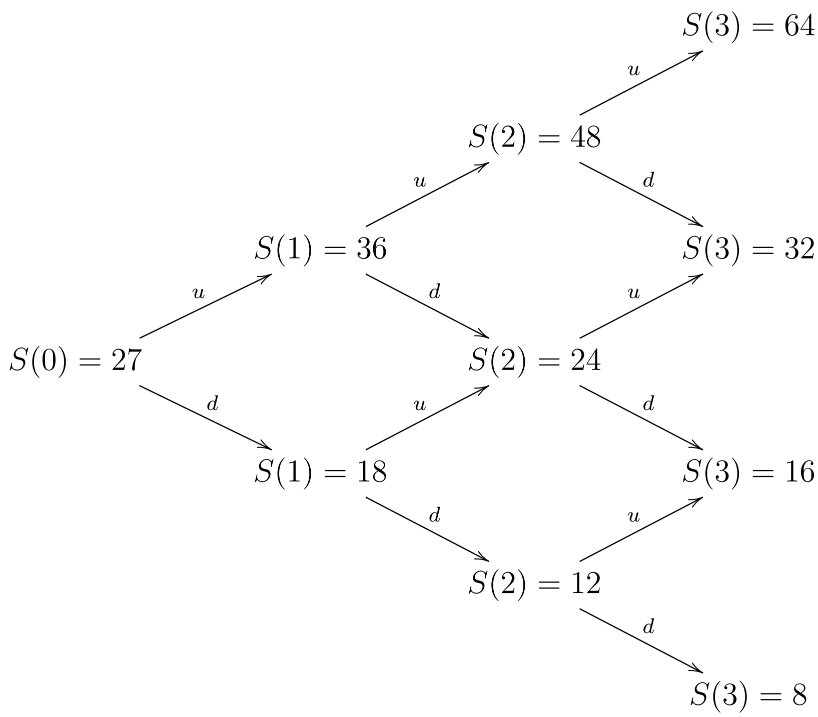
With the given values of the parameters $u, d, r$, we have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{1}{2} = q_d.$$

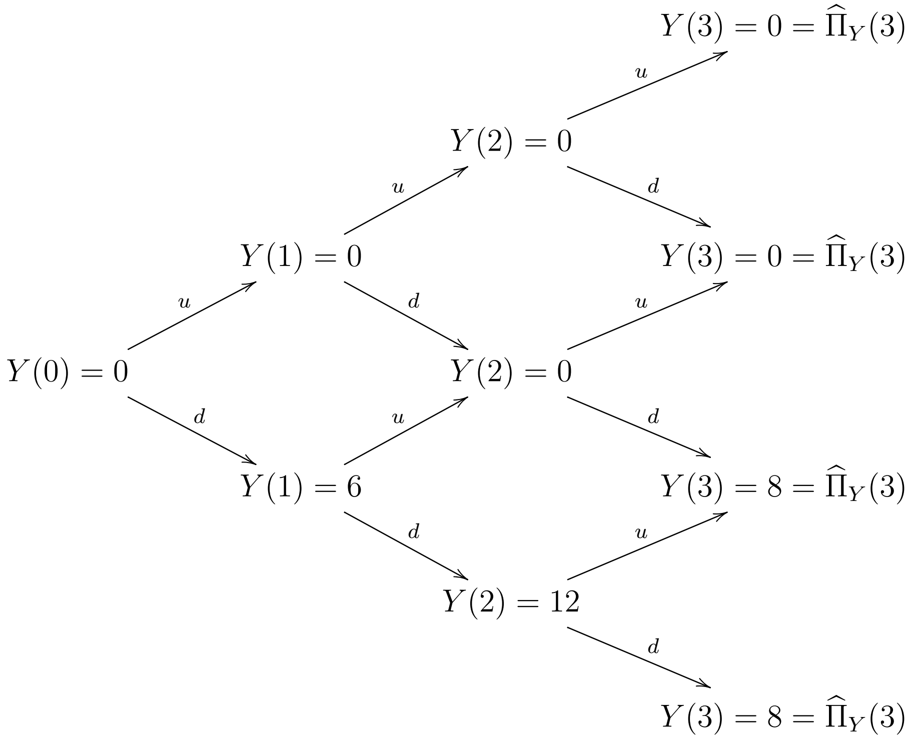
The fair price $\widehat{\Pi}_Y(t)$ of the American derivative satisfies

$$\begin{aligned}\widehat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u \widehat{\Pi}^u_Y(t+1) + q_d \widehat{\Pi}^d_Y(t+1))) \\ &= \max(Y(t), \frac{1}{2}(\widehat{\Pi}^u_Y(t+1) + \widehat{\Pi}^d_Y(t+1))),\end{aligned}$$

where $\widehat{\Pi}^u_Y(t)$ (resp. $\widehat{\Pi}^d_Y(t)$) is the price of the derivative at time $t$ assuming that the stock price goes up (resp. down) at time $t$. The diagram of the stock price is



to which there corresponds the following diagram for the intrinsic value:

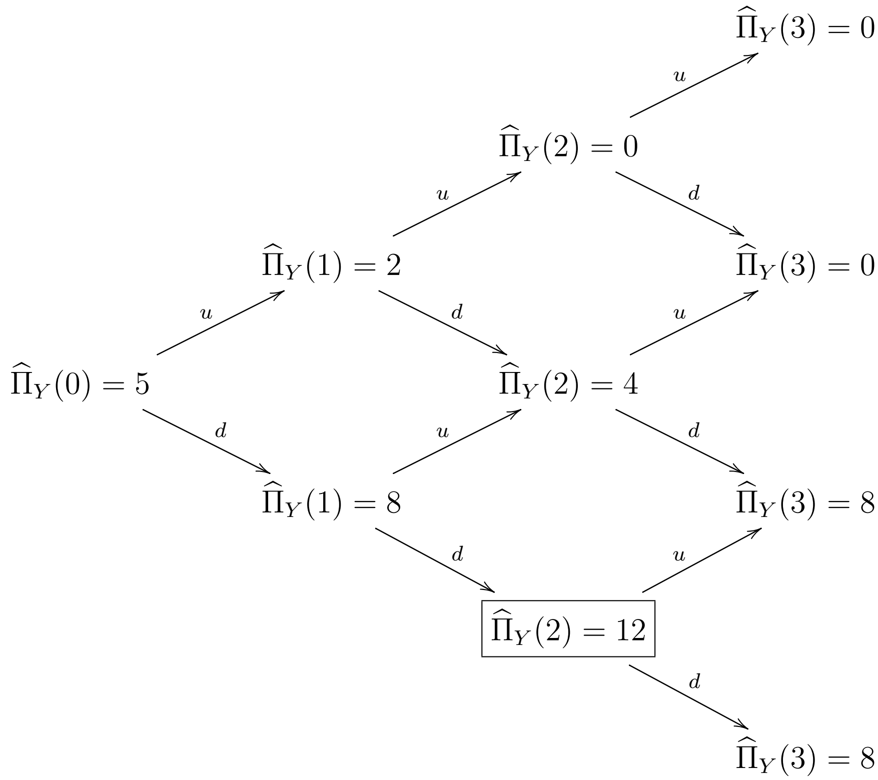



Therefore

$$S(2) = 48 \Rightarrow \widehat{\Pi}_Y(2) = 0, \quad S(2) = 24 \Rightarrow \widehat{\Pi}_Y(2) = 4, \quad S(2) = 12 \Rightarrow \widehat{\Pi}_Y(2) = 12$$

$$S(1) = 36 \Rightarrow \widehat{\Pi}_Y(1) = 2, \quad S(1) = 18 \Rightarrow \widehat{\Pi}_Y(1) = 8,$$

and $\widehat{\Pi}_Y(0) = 5$. We thereby obtained the following diagram for the price of the derivative:



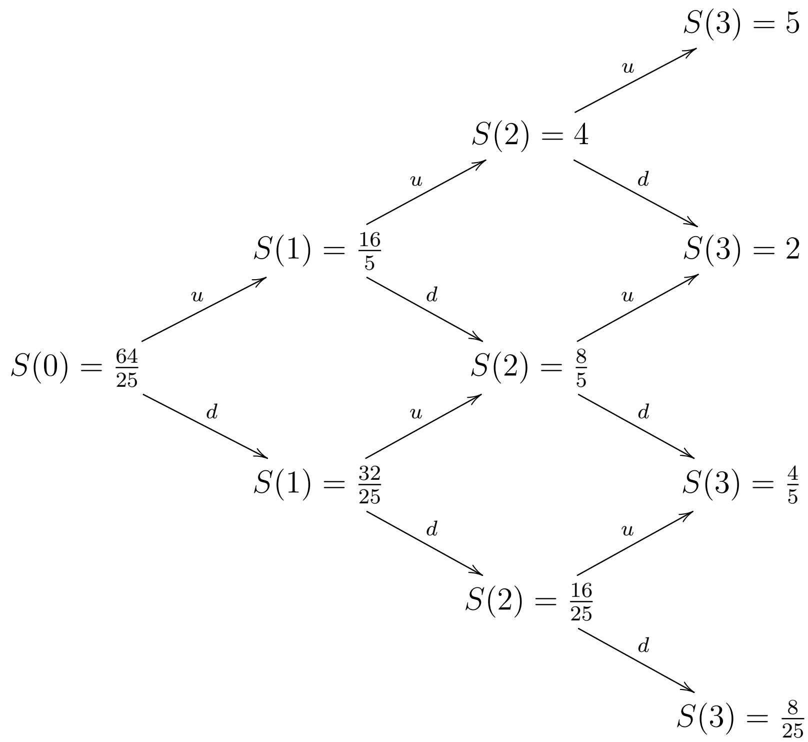

This completes the first part of the exercise. The only case in which the price of the derivative equals its intrinsic value prior to expiration is at time $t = 2$ when the price of the stock is $S(2) = 12$ (i.e., the stock price goes down in the first two steps). This is indicated in the previous diagram by putting the price of the derivative in a box. In this case, and only in this case, it is optimal to exercise the derivative prior to expiration. If the buyer does not exercise the derivative at the optimal moment, the writer can withdraw the amount

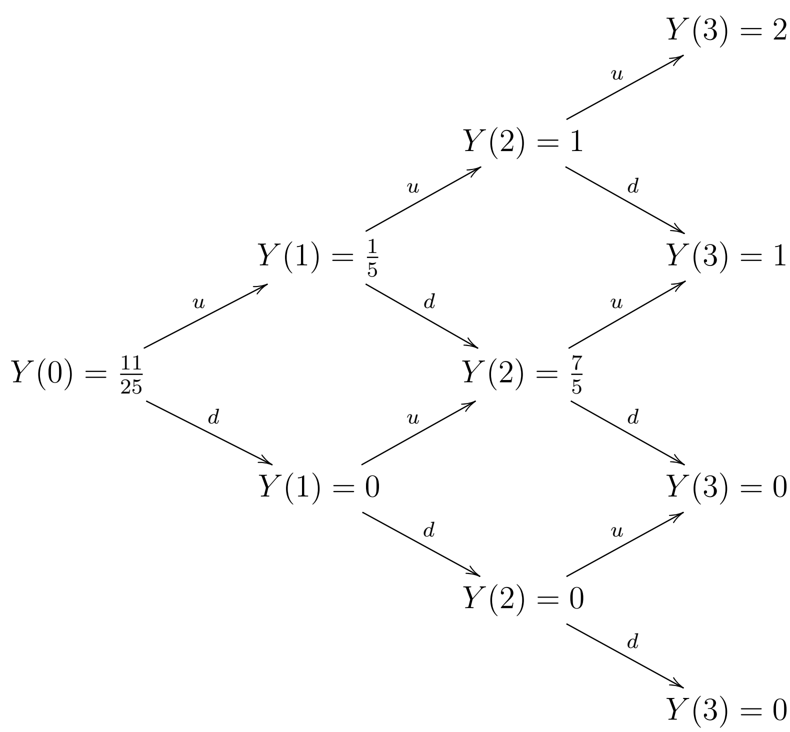
$$C(2) = \widehat{\Pi}_Y(2) - e^{-r}[q_u \widehat{\Pi}_Y(3)^u + q_d \widehat{\Pi}_Y^d(3)] = 12 - [\frac{1}{2}8 - \frac{1}{2}8] = 4.$$

Note that after withdrawing this cash, the value of the portfolio is 8, which is exactly what the writer needs to hedge the derivative!

## Exercise 4.5

The binomial trees for the stock price $S(t)$ and the intrinsic value $Y(t)$ are as follows.

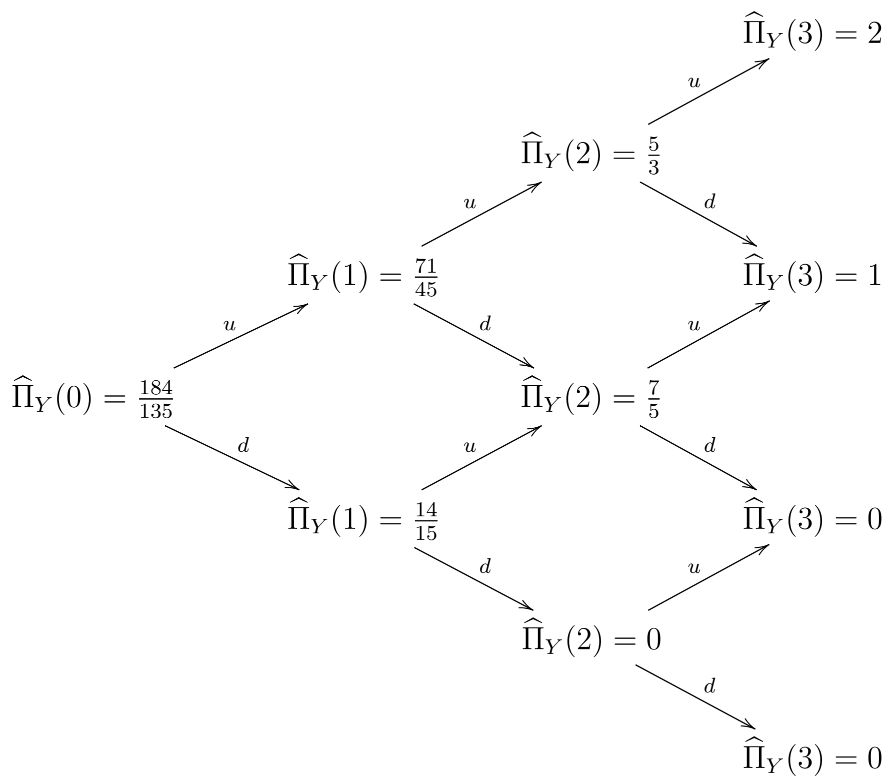


The binomial price of the American derivative is defined as

$$\widehat{\Pi}_Y(3) = Y(3), \quad \widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u\widehat{\Pi}_Y^u(t+1) + q_u\widehat{\Pi}_Y^u(t+1))], \ t = 0,1,2,$$

where $q_u = 2/3$ and $q_d = 1/3$. Applying the above formula one finds the following binomial tree for $\widehat{\Pi}_Y(t)$.




This concludes the first part of the exercise. The initial position on the stock in the hedging portfolio is

$$h_S(0) = h_S(1) = \frac{1}{S(0)}\frac{\widehat{\Pi}_Y^u(1) - \widehat{\Pi}_Y^d(1)}{e^u - e^d} = \frac{1}{\frac{64}{25}}\frac{\frac{71}{45} - \frac{14}{15}}{\frac{3}{4}} = \frac{145}{432},$$

which answers the second question. As to the cash flow, observe that the only optimal exercise time is $t = 2$ when $S(2) = 8/5$, as in this case, and only in this case, $\widehat{\Pi}_Y(t)$ and $Y(t)$ are equal. If the buyer does not exercise the derivative at this instance, the seller can withdraw the amount

$$C(2) = \widehat{\Pi}_Y(2) - e^{-r}(q_u\widehat{\Pi}_Y^u(t+1) + q_u\widehat{\Pi}_Y^u(t+1)) = \frac{7}{5} - \frac{2}{3} = \frac{11}{15}.$$

This answers the third question. The probability that the derivative is in the money at time $t$ is $\mathbb{P}(Y(t) > 0)$. We have

$$\mathbb{P}(Y(0) > 0) = 1, \quad \mathbb{P}(Y(1) > 0) = p,$$
$$\mathbb{P}(Y(2) > 0) = p^2 + 2p(1-p) = p(2-p), \quad \mathbb{P}(Y(3) > 0) = p^2 + 3p^2(1-p) = p^2(3-2p).$$

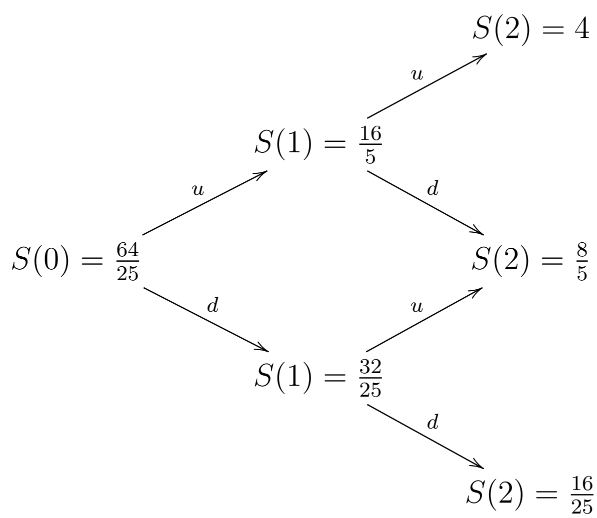
This answers the fourth question. Finally, the return for the buyer at time $t$ is positive if and only if $Y(t) > \Pi_Y(0)$ (the question is relevant because the buyer can exercise at any time). This happens only at time $t$ when $S(2) = 8/5$ (the optimal exercise time) and at maturity $t = 3$ when $S(3) = 5$. Hence we have

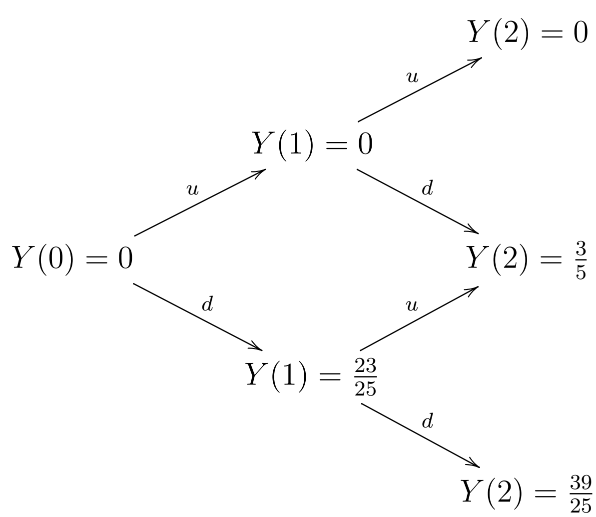
$$\mathbb{P}(R(0) > 0) = 0, \quad \mathbb{P}(R(1) > 0) = 0,$$
$$\mathbb{P}(R(2) > 0) = 2p(1-p) \quad \mathbb{P}(R(3) > 0) = p^3.$$

This completes the exercise.

## Exercise 4.6

The binomial tree for the stock price and for the intrinsic value $Y(t)$ of the American put are



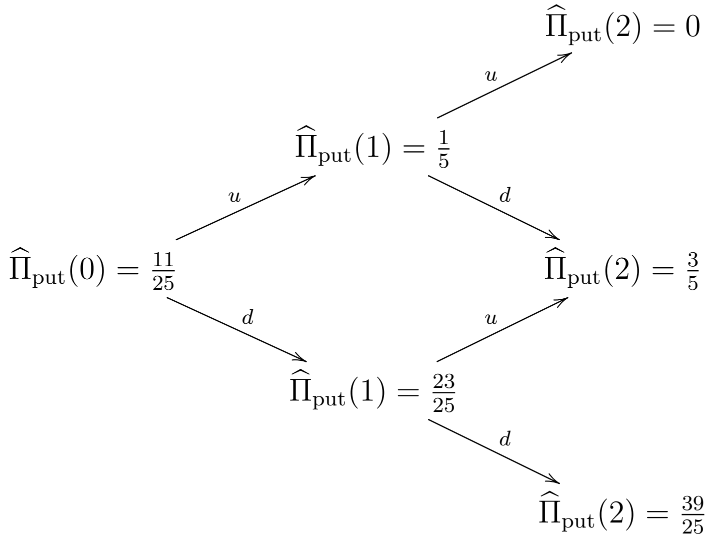
Let $\widehat{\Pi}_{\text{put}}(t)$ be the price at time $t \in \{0, 1, 2\}$ of the American put. We have the recurrence formula $\widehat{\Pi}_{\text{put}}(2) = Y(2)$ and

$$\widehat{\Pi}_{\text{put}}(t) = \max[Y(t), e^{-r}(q_u \widehat{\Pi}^u_{\text{put}}(t) + q_d \widehat{\Pi}^d_{\text{put}}(t))],$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{2}{3}, \quad q_d = 1 - q_u = \frac{1}{3}, \quad e^r = 1.$$

Hence the binomial tree for $\widehat{\Pi}_{\text{put}}(t)$ is




The only optimal exercise time prior to maturity is $t = 1$ when $S(1) = \frac{32}{25}$. This concludes the first part of the exercise. The compound option has maturity $T = 1$ and pay-off

$$Q = (\widehat{\Pi}_{\text{put}}(1) - \frac{8}{25})_+.$$

Since $\widehat{\Pi}_{\text{put}}(1)$ is a function of $S(1)$, then we can treat the compound option as a standard derivative on the stock. The compound option expires in the money if the stock price goes down at time $t = 1$ and out of the money otherwise. Hence the price of the compound option at time $t = 0$ is

$$\Pi_{\text{cp}}(0) = \frac{1}{3}(\frac{23}{25} - \frac{8}{25})_+ = \frac{1}{5}.$$

This answers the second question. As to the hedging portfolio, the compound option can be hedged by investing on the stock and the risk-free asset. The number of shares in the stock is

$$h_S = \frac{1}{S(0)} \frac{\Pi^u_{\text{cp}}(1) - \Pi^d_{\text{cp}}(1)}{e^u - e^d} = \frac{25}{64} \frac{0 - \frac{3}{5}}{\frac{5}{4} - \frac{1}{2}} = -\frac{5}{16}$$

The number of shares in the risk-free asset is obtained by solving the replicating equation at time $t = 0$:

$$h_S S(0) + h_B B(0) = \Pi_{\text{cp}}(0) \Rightarrow h_B = \frac{\Pi_{\text{cp}}(0) - h_S S(0)}{B(0)} = \frac{31}{25}.$$

This answers the third question. Finally we compute the expected return $\mathbb{E}[R]$ for the owner of the compound option as a function of $p \in (0,1)$. Clearly

$$R = -\frac{1}{5} \quad \text{with prob. } p,$$

which is the return when the stock price goes up at time $t = 1$. If the stock price goes down at time $t = 1$, the owner of the compound option will buy the American put for $K_1 = 8/25$. If the American put is exercised at this optimal exercise time, then the return will be

$$R = \frac{23}{25} - \frac{1}{5} - \frac{8}{25} = \frac{2}{5} \quad \text{with prob. } 1 - p.$$

Hence, if the American put is exercised at $t = 1$, the expected return is

$$\mathbb{E}[R] = -\frac{1}{5}p + \frac{2}{5}(1-p) = \frac{2}{5} - \frac{3}{5}p.$$

If the American put is exercised at $t = 2$, the expected return is

$$\mathbb{E}[R] = -\frac{1}{5}p + \big(\frac{3}{5} - \frac{8}{25} - \frac{1}{5}\big)p(1-p) + \big(\frac{39}{25} - \frac{8}{25} - \frac{1}{5}\big)(1-p)^2 = \frac{1}{25}(3p-2)(8p-13) = f(p).$$

Now, it is straightforward to verify that $f(p) > \frac{2}{5} - \frac{3}{5}p$ when $0 < p < 2/3$ and $f(p) < \frac{2}{5} - \frac{3}{5}p$ when $2/3 < p < 1$. Hence the strategy which maximizes the expected return for the compound option is: for $0 < p < 2/3$, the American put should *not* be exercised at time $t = 1$, while for $2/3 < p < 1$ the American put should be exercised at time $t = 1$. For $p = 2/3$ the two strategies lead to the same expected return. This answers the last question.

## Exercise 5.9

Let $A$ be an event that is resolved by both variables $X, Y$. This means that there exist $I, J \subseteq \mathbb{R}$ such that $A = \{X \in I\} = \{Y \in J\}$. Hence, using the independence of $X, Y$,

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J) = \mathbb{P}(A)\mathbb{P}(A) = \mathbb{P}(A)^2.$$

Therefore $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. In a finite probability space this implies $A = \emptyset$ or $A = \Omega$, respectively.

Now let $a, b$ be two deterministic constants. Note that, for all $I \subset \mathbb{R}$,

$$\mathbb{P}(a \in I) = \begin{cases} 1 & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases}$$

and similarly for $b$. Hence

$$\mathbb{P}(a \in I, b \in J) = \begin{cases} 1 & \text{if } a \in I \text{ and } b \in J \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(b \in J).$$

Finally we show that $X$ and $Y = g(X)$ are independent if and only if $Y$ is a deterministic constant. For the "if" part we use that

$$\mathbb{P}(a \in I, X \in J) = \begin{cases} \mathbb{P}(X \in J) & \text{if } a \in I \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}(a \in I)\mathbb{P}(X \in J).$$

For the "only if" part, let $z \in \mathbb{R}$ and $I = \{g(X) \leq z\} = \{X \in g^{-1}(-\infty, z]\}$. Then, using the independence of $X$ and $Y = g(X)$,

$$\begin{aligned} \mathbb{P}(g(X) \leq z) &= \mathbb{P}(g(X) \leq z, g(X) \leq z) = \mathbb{P}(X \in g^{-1}(-\infty, z], g(X) \leq z) \\ &= \mathbb{P}(X \in g^{-1}(-\infty, z])\mathbb{P}(g(X) \leq z) = \mathbb{P}(g(X) \leq z)\mathbb{P}(g(X) \leq z). \end{aligned}$$

Hence $\mathbb{P}(Y \leq z)$ is either 0 or 1, which implies that $Y$ is a deterministic constant.

## Exercise 5.10

The exercise asks to prove the following:

Let $X_1, X_2$ be independent random variables, $g : \mathbb{R} \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$. Then the random variables

$$Y = g(X_1), \quad Z = f(X_2)$$

are independent.

Given $I, J \subseteq \mathbb{R}$ we have $\{Y \in I\} = \{X_1 \in \{g \in I\}\}$ and $\{Z \in J\} = \{X_2 \in \{f \in J\}\}$. Hence, using the independence of $X_1, X_2$,

$$\begin{aligned} \mathbb{P}(Y \in I, Z \in J) &= \mathbb{P}(X_1 \in \{g \in I\}, X_2 \in \{f \in J\}) \\ &= \mathbb{P}(X_1 \in \{g \in I\})\mathbb{P}(X_2 \in \{f \in J\}) = \mathbb{P}(Y \in I)\mathbb{P}(Z \in J). \end{aligned}$$

## Exercise 5.11

The statement $X, Y$ independent $\Rightarrow$ $X, Y$ uncorrelated holds for random variables on general probability spaces, but here we are only concerned with finite probability spaces. In particular, $X$ can only take a finite number of values $x_1, \ldots x_N$ and $Y$ a finite number of values $y_1, \ldots y_M$. Letting $A_i = \{X = x_i\}$, $B_j = \{Y = y_j\}$, $i = 1, \ldots N$, $j = 1, \ldots M$, and denoting $\mathbb{I}_A$ the indicator function of the set $A$, we have

$$X = \sum_{i=1}^{N} x_i \mathbb{I}_{A_i}, \quad Y = \sum_{j=1}^{M} y_j \mathbb{I}_{B_j}.$$

Hence

$$XY = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{I}_{A_i} \mathbb{I}_{B_j} = \sum_{i=1}^{N} \sum_{j=1}^{M} x_i y_j \mathbb{I}_{A_i \cap B_j}$$

Hence, by the linearity of the expectation, and the assumed independence of $X, Y$,

$$
\begin{aligned}
\mathbb{E}[XY] &= \sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j \mathbb{E}[\mathbb{I}_{A_i\cap B_j}] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j \mathbb{P}(A_i \cap B_j) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{M} x_i y_j \mathbb{P}(A_i)\mathbb{P}(B_j) \\
&= \sum_{i=1}^{N} x_i \mathbb{P}(A_i) \sum_{j=1}^{M} \mathbb{P}(B_j) = \mathbb{E}[X]\mathbb{E}[Y].
\end{aligned}
$$

As an example of uncorrelated, but not independent, random variables $X, Y$, consider

$$
X = \begin{cases} -1 & \text{with prob. } 1/3 \\ 0 & \text{with prob. } 1/3 \\ 1 & \text{with prob. } 1/3 \end{cases} \qquad Y = X^2.
$$

The random variables $X, Y$ are not independent, since $Y$ is not a deterministic constant (see Exercise 5.9 above). Moreover $XY = X^3 = X$ and thus $\mathbb{E}[XY] = \mathbb{E}[X^3] = \mathbb{E}[X] = 0$. Since $\mathbb{E}[X]\mathbb{E}[Y] = 0$, then $\mathrm{Cov}(X, Y) = 0$, i.e., the two random variables are uncorrelated. As to (5.18), we write

$$
\mathrm{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[(X+Y)]^2 = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]),
$$

by which the claim follows.

## Exercise 5.12

To prove the inequality we first we notice that

$$
\mathrm{Var}[\alpha X] = \mathbb{E}[\alpha^2 X^2] - \mathbb{E}[\alpha X]^2 = \alpha^2 \mathbb{E}[X^2] - \alpha^2 \mathbb{E}[X]^2 = \alpha^2 \mathrm{Var}[X],
$$

$$
\mathrm{Cov}(\alpha X, Y) = \mathbb{E}[\alpha XY] - \mathbb{E}[\alpha X]\mathbb{E}[Y] = \alpha \mathrm{Cov}(X, Y)
$$

and

$$
\begin{aligned}
\mathrm{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY] \\
&- \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X, Y).
\end{aligned}
$$

Hence letting $a \in \mathbb{R}$ we have

$$
\mathrm{Var}[Y - aX] = a^2 \mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X, Y).
$$

Since the variance of a random variable is always non-negative, the parabola

$$y(a) = a^2\mathrm{Var}[X] + \mathrm{Var}[Y] - 2a\mathrm{Cov}(X, Y)$$

must always lie above the $a$-axis, or touch it at one single point $a = a_0$. Hence

$$\mathrm{Cov}(X, Y)^2 - \mathrm{Var}[X]\mathrm{Var}[Y] \leq 0,$$

which proves (5.19). Moreover $\mathrm{Cov}(X, Y)^2 = \mathrm{Var}[X]\mathrm{Var}[Y]$ if and only if there exists $a_0$ such that $\mathrm{Var}[-a_0X + Y] = 0$, i.e., $Y = a_0X + b_0$, for some constant $b_0$. Note that $a_0 \neq 0$, otherwise $Y$ is a deterministic constant. Substituting in the definition of covariance, we see that $\mathrm{Cov}(X, a_0X + b_0) = a_0\mathrm{Var}[X]$. Hence if the right inequality in (5.19) is an equality we have

$$a_0\mathrm{Var}[X] = \sqrt{\mathrm{Var}[X]\mathrm{Var}[a_0X + b_0])}, \quad \text{i.e.,}\, a_0\mathrm{Var}[X] = |a_0|\mathrm{Var}[X],$$

and thus $a_0 > 0$. Similarly one shows that if the left inequality becomes an equality then $a_0 < 0$.

## Exercise 5.17

By linearity of the expectation,

$$\mathbb{E}[W_n(t)] = \frac{1}{\sqrt{n}}\mathbb{E}[M_{[nt]}] = 0,$$

where we used that $\mathbb{E}[X_k] = \mathbb{E}[M_k] = 0$. Since $\mathrm{Var}[M_k] = k$, we obtain

$$\mathrm{Var}[W_n(t)] = \frac{[nt]}{n}.$$

Since $nt \sim [nt]$, as $n \to \infty$, then $\lim_{n\to\infty} \mathrm{Var}[W_n(t)] = t$. As to the covariance of $W_n(t)$ and $W_n(s)$ for $s \neq t$, we compute

$$\begin{aligned}\mathrm{Cov}[W_n(t), W_n(s)] &= \mathbb{E}[W_n(t)W_n(s)] - \mathbb{E}[W_n(t)]\mathbb{E}[W_n(s)] = \mathbb{E}[W_n(t)W_n(s)] \\ &= \mathbb{E}\left[\frac{1}{\sqrt{n}}M_{[nt]}\frac{1}{\sqrt{n}}M_{[ns]}\right] = \frac{1}{n}\mathbb{E}[M_{[nt]}M_{[ns]}]. \qquad \text{(B.6)}\end{aligned}$$

Assume $t > s$ (a similar argument applies to the case $t < s$). If $[nt] = [ns]$ we have $\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathrm{Var}[M_{[ns]}] = [ns]$. If $[nt] \geq 1 + [ns]$ we have

$$\mathbb{E}[M_{[nt]}M_{[ns]}] = \mathbb{E}[(M_{[nt]} - M_{[ns]})M_{[ns]}] + \mathbb{E}[M_{[ns]}^2] = \mathbb{E}[M_{[nt]} - M_{[ns]}]\mathbb{E}[M_{[ns]}] + \mathrm{Var}[M_{[ns]}] = [ns],$$

where we used that the increment $M_{[nt]} - M_{[ns]}$ is independent of $M_{[ns]}$. Replacing into (B.6) we obtain

$$\mathrm{Cov}[W_n(t), W_n(s)] = \frac{[ns]}{n}.$$

It follows that $\lim_{n\to\infty} \mathrm{Cov}[W_n(t), W_n(s)] = s$.

## Exercise 5.19

The second formula follows by the first one using that $e^u q + e^d(1-q) = e^r$ (or by letting $t = N$ in (5.28)). To prove the first formula we use

$$\mathbb{E}_p[S(N)] = \mathbb{E}_p[S(0)\exp(X_1 + \cdots + X_N)] = S(0)\mathbb{E}_p[Y],$$

where $Y$ is the random variable

$$Y(\omega) = \exp(X_1(\omega) + \cdots + X_N(\omega)) = \exp(uN_H(\omega) + dN_T(\omega)), \quad \omega \in \Omega_N.$$

Hence, using $N_T(\omega) = N - N_H(\omega)$ and (5.10) we obtain

$$\begin{aligned}\mathbb{E}_p[S(N)] &= S(0) \sum_{\omega\in\Omega_N} e^{(uN_H(\omega)+dN_T(\omega))} p^{N_H(\omega)}(1-p)^{N_T(\omega)} \\ &= S(0)e^{Nd}(1-p)^N \sum_{\omega\in\Omega_N} e^{(u-d)N_H(\omega)} \left(\frac{p}{(1-p)}\right)^{N_H(\omega)}.\end{aligned}$$

Now we use that for $k = 0, \dots, N$ there exist exactly $\binom{N}{k}$ sample points $\omega \in \Omega_N$ such that $N_H(\omega) = k$. Hence we can write

$$\mathbb{E}_p[S(N)] = S(0)e^{Nd}(1-p)^N \sum_{k=0}^{N} \binom{N}{k} \left(\frac{e^u p}{e^d(1-p)}\right)^k.$$

By the binomial theorem, $(1+a)^N = \sum_{k=0}^N \binom{N}{k} a^k$, hence

$$\mathbb{E}_p[S(N)] = S(0)e^{Nd}(1-p)^N \left(1 + \frac{e^u p}{e^d(1-p)}\right)^N = S(0)(e^d(1-p) + e^u p)^N.$$

Similarly one finds that $\mathrm{Var}_p[S(N)] = S(0)^2[(e^{2u}p + e^{2d}(1-p))^N - (e^u p + e^d(1-p))^{2N}]$.

## Exercise 5.20

By the risk neutral pricing formula (5.35) at time $t = 0$ we have $AC(0) = e^{-rN}\mathbb{E}_q[Y_{\mathrm{call}}(x)]$ and similarly for the Asian put. Thus

$$AC(0) - AP(0) = \mathbb{E}_q[Y_{\mathrm{call}} - Y_{\mathrm{put}}]$$

Using

$$Y_{\mathrm{call}} - Y_{\mathrm{put}} = \left[\left(\frac{1}{N+1}\sum_{t=0}^N S(t)\right) - K\right]_+ - \left[K - \left(\frac{1}{N+1}\sum_{t=0}^N S(t)\right)\right]_+ = \left(\frac{1}{N+1}\sum_{t=0}^N S(t)\right) - K$$

we find

$$AC(0) - AP(0) = \frac{e^{-rN}}{N+1}\left(\sum_{t=0}^N \mathbb{E}_q[S(t)]\right) - Ke^{-rN}\mathbb{E}_q[1],$$

where $\mathbb{E}_q[\cdot]$ denotes the expectation in the risk-neutral measure. As $\mathbb{E}_q[1] = 1$ and $\mathbb{E}_q[S(t)] = S(0)e^{rt}$, we obtain

$$AC(0) - AP(0) = e^{-rN}\Big[\frac{S(0)}{N+1}\left(\sum_{t=0}^{N} e^{rt}\right) - K\Big].$$

Using the formula in the HINT concludes the exercise.

## Exercise 5.27

As $\mathbb{E}[W(t)] = 0$ for all $t \geq 0$,

$$\mathrm{Cov}[W(s), W(t)] = \mathbb{E}[W(s)W(t)].$$

Assume $t > s$ (for $t < s$ the argument is identical). Using that the increments $W(t) - W(s)$ and $W(s) = W(s) - W(0)$ are independent we have

$$\begin{aligned}\mathbb{E}[W(s)W(t)] &= \mathbb{E}[W(s)(W(t) - W(s))] + \mathbb{E}[W(s)^2] \\ &= \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] + \mathrm{Var}[W(s)] = \mathrm{Var}[W(s)] = s.\end{aligned}$$

## Exercise 6.9

Recall that

$$C(t, x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{B.7}$$

where

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{B.8}$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$ is the standard normal distribution. As $\sigma \to 0^+$ we have $d_1 \to d_2$ and

$$d_2 \sim \frac{1}{\sqrt{\tau}}(\log\frac{x}{K} + r\tau)\sigma^{-1}.$$

Hence

$$\begin{aligned} &d_2 \to +\infty, \quad \text{if } x > Ke^{-r\tau}, \\ &d_2 \to -\infty, \quad \text{if } x < Ke^{-r\tau}, \\ &d_2 \to 0, \quad \text{if } x = Ke^{-r\tau}, \end{aligned}$$

Thus

$$\begin{aligned} \lim_{\sigma\to 0^+}\Phi(d_1) &= \lim_{\sigma\to 0^+}\Phi(d_2) = 1, \quad \text{if } x > Ke^{-r\tau}, \\ \lim_{\sigma\to 0^+}\Phi(d_1) &= \lim_{\sigma\to 0^+}\Phi(d_2) = 0, \quad \text{if } x < Ke^{-r\tau}, \\ \lim_{\sigma\to 0^+}\Phi(d_1) &= \lim_{\sigma\to 0^+}\Phi(d_2) = \Phi(0), \quad \text{if } x = Ke^{-r\tau}. \end{aligned}$$

It follows that

$$\lim_{\sigma\to 0^+} C(t,x) = x - Ke^{-r\tau} \quad \text{if } x > Ke^{-r\tau},$$
$$\lim_{\sigma\to 0^+} C(t,x) = 0, \quad \text{if } x \le Ke^{-r\tau},$$

i.e., $\lim_{\sigma\to 0^+} C(t,x) = (x - Ke^{-r\tau})_+$. For $\sigma \to +\infty$ we have $d_2 \to -\infty$ and $d_1 \to +\infty$, hence $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 0$. Thus $C(t,x) \to x$ as $\sigma \to +\infty$. As $K \to 0^+$, both $d_1$ and $d_2$ diverge to $+\infty$, hence

$$\lim_{K\to 0^+} C(t,x) = x.$$

For $K \to +\infty$, $d_1, d_2$ diverge to $-\infty$. Hence the first term in $C(t,x)$ converges to zero. As the first term in $C(t,x)$ always dominates the second term (since $C(t,x) > 0$), then the second term also goes to zero and thus

$$\lim_{K\to +\infty} C(t,x) = 0.$$

For $T \to +\infty$ we obtain

$$\lim_{T\to +\infty} C(t,x) = x.$$

Finally, for $x \to 0^+$, both $d_1, d_2$ diverge to $-\infty$ and thus

$$\lim_{x\to 0^+} C(t,x) = 0.$$

To compute the limits for put options we use the put-call parity:

$$C(t,x) - P(t,x) = x - Ke^{-r\tau},$$

by which it follows that

$$\lim_{\sigma\to 0^+} P(t,x) = (Ke^{-r\tau} - x)_+, \quad \lim_{\sigma\to +\infty} P(t,x) = Ke^{-r\tau}$$
$$\lim_{K\to 0^+} P(t,x) = 0, \quad \lim_{K\to +\infty} P(t,x) = +\infty,$$
$$\lim_{T\to +\infty} P(t,x) = 0, \quad \lim_{x\to 0^+} P(t,x) = Ke^{-r\tau}.$$

## Exercise 6.13

The pay-off function is $g(z) = k + z\log z$. Hence the Black-Scholes price of the derivative is $\Pi_Y(t) = v(t, S(t))$, where

$$\begin{aligned}
v(t,x) &= e^{-r\tau} \int_{\mathbb{R}} g\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\tau+\sigma\sqrt{\tau}y}\right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
&= e^{-r\tau} \int_{\mathbb{R}} \left(k + xe^{(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}y}(\log x + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y)\right) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
&= ke^{-r\tau} + x\log x \int_{\mathbb{R}} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2} \frac{dy}{\sqrt{2\pi}} \\
&\quad + x(r - \frac{\sigma^2}{2})\tau \int_{\mathbb{R}} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2} \frac{dy}{\sqrt{2\pi}} + x\sigma\sqrt{\tau} \int_{\mathbb{R}} ye^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2} \frac{dy}{\sqrt{2\pi}}
\end{aligned}$$

Using that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\frac{dy}{\sqrt{2\pi}} = 1, \quad \int_{\mathbb{R}} ye^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\frac{dy}{\sqrt{2\pi}} = \sigma\sqrt{\tau},$$

we obtain

$$v(t,x) = ke^{-r\tau} + x\log x + x(r + \frac{\sigma^2}{2})\tau.$$

Hence

$$\Pi_Y(t) = ke^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau.$$

This completes the first part of the exercise. The number of shares of the stock in the hedging portfolio is given by

$$h_S(t) = \Delta(t, S(t)),$$

where $\Delta(t,x) = \frac{\partial v}{\partial x} = \log x + 1 + (r + \frac{\sigma^2}{2})\tau$. Hence

$$h_S(t) = 1 + (r + \frac{\sigma^2}{2})\tau + \log S(t).$$

The number of shares of the risk-free asset is obtained by using that

$$\Pi_Y(t) = h_S(t)S(t) + B(t)h_B(t),$$

hence

$$\begin{aligned} h_B(t) &= \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) \\ &= e^{-rt}(ke^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t) - S(t)(r + \frac{\sigma^2}{2})\tau - S(t)\log S(t)) \\ &= ke^{-rT} - S(t)e^{-rt}. \end{aligned}$$

This completes the second part of the exercise. To compute the probability that $Y > 0$, we first observe that the pay-off function $g(z)$ has a minimum at $z = e^{-1}$ and we have $g(e^{-1}) = k - e^{-1}$. Hence if $k \geq e^{-1}$, the derivative has probability 1 to expire in the money. If $k < e^{-1}$, there exist $a < b$ such that

$$g(z) > 0 \quad \text{if and only if} \quad 0 < z < a \text{ or } z > b.$$

Hence for $k < e^{-1}$ we have

$$\mathbb{P}(Y > 0) = \mathbb{P}(S(T) < a) + \mathbb{P}(S(T) > b).$$

Since $S(T) = S(0)e^{\alpha T + \sigma\sqrt{T}G}$, with $G \in N(0,1)$, then

$$S(T) < a \Leftrightarrow G < -\frac{\log\frac{S(0)}{a} + \alpha T}{\sigma\sqrt{T}} := -A, \quad S(t) > b \Leftrightarrow G > -\frac{\log\frac{S(0)}{b} + \alpha T}{\sigma\sqrt{T}} := -B.$$

Thus

$$\begin{aligned} \mathbb{P}(Y > 0) = \mathbb{P}(G < -A) + \mathbb{P}(G > -B) &= \int_{-\infty}^{-A} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} + \int_{-B}^{+\infty} e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} \\ &= \Phi(-A) + 1 - \Phi(-B) = 1 - \Phi(A) + \Phi(B). \end{aligned}$$

This completes the solution of the third part of the exercise.

## Exercise 6.14

The pay-off function is $g(z) = z(z-K)$; the Black-Scholes price is given by $\Pi_Y(t) = v(t, S(t))$, where

$$
\begin{aligned}
v(t,x) &= e^{-r\tau}\int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}y})e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} \\
&= e^{-r\tau}\int_{\mathbb{R}} \left[xe^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}y}\right]\left[se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}y} - K\right]e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}} \\
&= x\left[xe^{(r+\sigma^2)\tau} - K\right].
\end{aligned}
$$

This solves the first part of the exercise. The number of shares of the underlying stock in the hedging portfolio is given by $h_S(t) = \Delta(t, S(t))$, where

$$\Delta(t,x) = \frac{\partial v}{\partial x} = 2xe^{(r+\sigma^2)\tau} - K.$$

The number of shares of the risk-free is obtained using

$$\Pi_Y(t) = h_S(t)S(t) + h_B(t)B(t) \Rightarrow h_B(t) = \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) = -\frac{e^{-rt}}{B(0)}e^{(r+\sigma^2)\tau}S(t)^2.$$

This completes the second part of the exercise. The relative return of the portfolio is

$$R = \frac{\Pi_Y(T)}{\Pi_Y(0)} - 1.$$

Using $\Pi_Y(T) = S(T)(S(T)-K)$ and $\Pi_Y(0) = K^2(e^{(r+\sigma^2)T} - 1)$, we obtain

$$\mathbb{E}[R] = \frac{\mathbb{E}[S(T)(S(T)-K)]}{K^2(e^{(r+\sigma^2)T}-1)} - 1.$$

Writing the geometric Brownian motion at time $T$ as

$$S(T) = S(0)e^{(\mu-\frac{\sigma^2}{2})T+\sigma W(T)} = Ke^{(\mu-\frac{\sigma^2}{2})T+\sigma\sqrt{T}G},$$

where $G = W(T)/\sqrt{T} \in N(0,1)$, we get

$$
\begin{aligned}
\mathbb{E}[S(T)^2] &= K^2e^{(2\mu-\sigma^2)T}\int_{\mathbb{R}} e^{2\sigma\sqrt{T}y-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} = K^2e^{(2\mu+\sigma^2)T}, \\
\mathbb{E}[S(T)] &= Ke^{(\mu-\frac{\sigma^2}{2})T}\int_{\mathbb{R}} e^{\sigma\sqrt{T}y-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}} = Ke^{\mu T}.
\end{aligned}
$$

Therefore

$$\mathbb{E}[S(T)(S(T)-K)] = \mathbb{E}[S(T)^2] - K\mathbb{E}[S(T)] = K^2e^{\mu T}(e^{(\mu+\sigma^2)T} - 1).$$

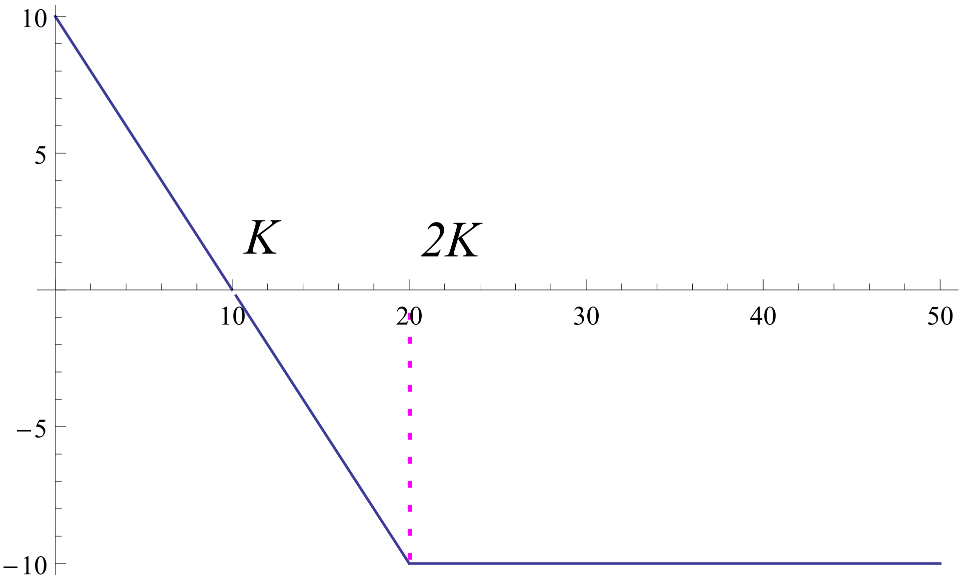
We conclude that the expected relative return is given by

$$\mathbb{E}[R] = \frac{e^{\mu T}(e^{(\mu+\sigma^2)T}-1)}{e^{(r+\sigma^2)T}-1} - 1.$$

This completes the third part of the exercise.

## Exercise 6.15

The graph of the pay-off looks like in the following picture (the numbers on the axes are irrelevant).




By this picture one can see that

$$g(x) = (K - x)_+ - (x - K)_+ + (x - 2K)_+ = g_1(x) + g_2(x) + g_3(x).$$

As the Black-Scholes price is linear in the pay-off function, the Black-Scholes price of the derivative is the sum of the Black-Scholes price of the derivatives with pay-off functions $g_1, g_2, g_3$, hence

$$\begin{aligned}\Pi_Y(t) &= P(t, S(t), K, T) - C(t, S(t), K, T) + C(t, S(t), 2K, T) \\ &= Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T),\end{aligned}$$

where for the second equality we used the put-call parity. Here $C(t, S(t), K, T)$ denotes the Black-Scholes price of the European call with strike $K$ and maturity $T$. Hence

$$C(t, S(t), 2K, T) = S(t)\Phi(\widetilde{d}_1) - 2Ke^{-r\tau}\Phi(\widetilde{d}_2),$$

where

$$\widetilde{d}_1 = \frac{\log \frac{S(t)}{2K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad \widetilde{d}_2 = \frac{\log \frac{S(t)}{2K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

This completes the second part of the exercise. For the last part, denote

$$f(K) = Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T)$$

the price of the derivative as a function of $K$. We want to prove that there exists $K_* > 0$ such that $f(K_*) = 0$. First we observe that $f(K) > 0$ for $K > S(t)e^{r\tau}$. Hence, if we prove that $f(K)$ can also take negative values, then, since $f$ is continuous, there must exist $K_*$ such that $f(K_*) = 0$. To this purpose define $K_0$ such that $\widetilde{d}_2 = 0$, that is

$$K_0 = \frac{1}{2} S(t) e^{(r - \frac{\sigma^2}{2})\tau}.$$

For this strike price we have $\Phi(\widetilde{d}_2) = 1/2$ and so the function $f$ evaluated at $K_0$ is

$$f(K_0) = (\Phi(\widetilde{d}_1) - 1) S(t).$$

As $\Phi(\widetilde{d}_1) < 1$, we find $f(K_0) < 0$. Hence there exists $K_0 < K_* < S(t)e^{r\tau}$ such that $f(K_*) = 0$. The value $K_*$ is the fair value of the strike price for the derivative. In fact, as both the buyer and the seller of this derivative can loose money at maturity, the fair price of the derivative should be zero. This concludes the solution of the exercise.

## Exercise 6.19

Letting $a = 1 - e^{-rT}$, the price at time $t = 0$ of a European derivative with pay-off function $g(x) = (x - S(0))_+$ is

$$\begin{aligned}
\Pi_Y(0) &= e^{-rT} \int_{\mathbb{R}^3} g((1-a)S(0)e^{(r-\sigma^2/2)T + \sigma\sqrt{T}y}) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT} S(0) \int_{\mathbb{R}^3} (e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}y} - 1)_+ e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT} S(0) \int_{\sigma\sqrt{T}/2}^{+\infty} \left( e^{-\frac{1}{2}(y - \sigma\sqrt{T})^2} - e^{\frac{y^2}{2}} \right) \frac{dy}{\sqrt{2\pi}} \\
&= e^{-rT} S(0) [\Phi(\frac{\sigma\sqrt{T}}{2}) - \Phi(-\frac{\sigma\sqrt{T}}{2})] \\
&= S(0) e^{-rT} (2\Phi(\frac{\sigma\sqrt{T}}{2}) - 1)
\end{aligned}$$

where $\Phi$ is the standard normal cumulative distribution. The value of the given portfolio at times $t = 0, T$ is

$$V(T) = -g(S(T)) - S(0)e^{-rT}e^{rT} = -(S(T) - S(0))_+ - S(0)$$

$$V(0) = -\Pi_Y(0) - S(0)e^{-rT} = -2S(0)e^{-rT}\Phi(\frac{\sigma\sqrt{T}}{2})$$

Hence $R(T) = V(T) - V(0)$ is given by

$$R(T) = \begin{cases} -S(T) + 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2) & \text{if } S(T) > S(0) \\ S(0)[2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1] & \text{if } S(T) < S(0) \end{cases}$$

In particular, for $S(T) > S(0)$ we have

$$R(T) < S(0)[2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1], \quad \text{for } S(T) > S(0).$$

It follows that for $2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1 \le 0$, the portfolio return is always negative. For $2e^{-rT}\Phi(\sigma\sqrt{T}/2) - 1 > 0$ we have

$$\begin{aligned}\mathbb{P}(R(T) > 0) &= \mathbb{P}(S(T) > S(0)) + \mathbb{P}(S(0) < S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2)) \\ &= \mathbb{P}(S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2))\end{aligned}$$

As

$$S(T) = (1-a)S(0)e^{(r-\sigma^2/2)T+\sigma\sqrt{T}G} = S(0)^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}G},$$

where $G$ is a standard normal random variable, then

$$S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2) \Leftrightarrow G \le \delta$$

where

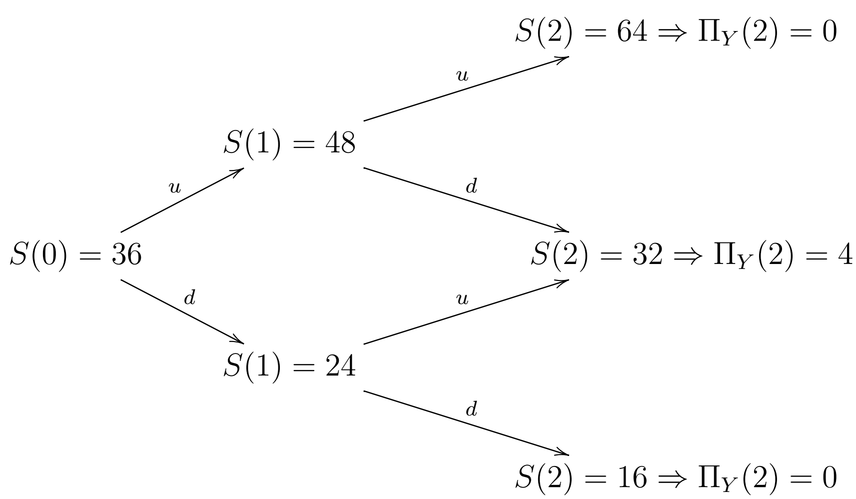
$$\delta = \frac{(\frac{\sigma^2}{2} - r)T + \log(2\Phi(\sigma\sqrt{T}/2))}{\sigma\sqrt{T}}.$$

Hence

$$\mathbb{P}(S(T) < 2S(0)e^{-rT}\Phi(\sigma\sqrt{T}/2)) = \mathbb{P}(G \le \delta) = \Phi(\delta).$$

## Exercise A.2

The diagram of the stock price is




where we also indicated the value of the derivative at the expiration date (which is equal to the pay-off). The parameters of the binomial model are

$$q_u = q_d = \frac{1}{2}, \quad r = 0.$$

To compute the price of the derivative at the times $t \in \{0, 1\}$ we use the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u\Pi_Y^u(t+1) + q_d\Pi_Y^d(t+1)) = \frac{1}{2}(\Pi_Y^u(t+1) + \Pi_Y^d(t+1)), \quad t \in \{0, 1\}.$$

Hence at time $t = 1$ we have

$$S(1) = 48 \Rightarrow \Pi_Y(1) = \frac{1}{2}(0+4) = 2$$
$$S(1) = 24 \Rightarrow \Pi_Y(1) = \frac{1}{2}(4+0) = 2$$

and at time $t = 0$ we have

$$\Pi_Y(0) = \frac{1}{2}(2+2) = 2$$

This concludes the first part of the exercise. Next, let $a_1$ be the number of shares of the stock and $a_2$ the number of shares of the derivative in the Markowitz portfolio and let

$$\pi_1 = \frac{a_1 S(0)}{K}, \qquad \pi_2 = \frac{a_2 \Pi_Y(0)}{K}.$$

Then we have the formula

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{2\theta} C^{-1} \begin{pmatrix} \mu_1 - \rho \\ \mu_2 - \rho \end{pmatrix} \tag{B.9}$$

where the various symbols have the following meaning: $\theta$ is the risk aversion of the investor (which is 1/36 in this exercise), $\mu_i = \mathbb{E}[R_i]$, $R_1, R_2$ are the relative returns of the stock and the derivative, $\rho = e^{2r} - 1$ (which is 1/9 in this exercise) and $C^{-1}$ is the inverse of the covariant matrix $C_{ij} = C(R_i, R_j)$. Hence we first compute the random variables $R_1, R_2$, which are defined as

$$R_1 = \frac{S(2) - S(0)}{S(0)}, \quad R_2 = \frac{\Pi_Y(2) - \Pi_Y(0)}{\Pi_Y(0)}.$$

Since $p = 1/2$, then

$$S(2) = \begin{cases} 64 & \text{with probability } 1/4 \\ 32 & \text{with probability } 1/2 \\ 16 & \text{with probability } 1/4 \end{cases}$$

and similarly

$$\Pi_Y(2) = \begin{cases} 0 & \text{with probability } 1/4 \\ 4 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/4 \end{cases}$$

Note that $S(2)$ and $\Pi_Y(2)$ are not independent variables! In fact, the price of the derivative at time 2 is uniquely determined by the price of the stock at time 2. Using the above we find

$$R_1 = \begin{cases} 7/9 & \text{with probability } 1/4 \\ -1/9 & \text{with probability } 1/2 \\ -5/9 & \text{with probability } 1/4 \end{cases}$$

$$R_2 = \begin{cases} -1 & \text{with probability } 1/4 \\ 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/4 \end{cases}$$

Hence

$$\mu_1 = \mathbb{E}[R_1] = \frac{7}{9} \cdot \frac{1}{4} - \frac{1}{9} \cdot \frac{1}{2} - \frac{5}{9} \cdot \frac{1}{4} = 0$$

and similarly $\mu_2 = \mathbb{E}[R_2] = 0$. Next we compute the covariance matrix. We have

$$C_{11} = \mathrm{Cov}(R_1, R_1) = \mathrm{Var}[R_1] = \mathbb{E}[R_1^2] = \frac{49}{81} \cdot \frac{1}{4} + \frac{1}{81} \cdot \frac{1}{2} + \frac{25}{81} \cdot \frac{1}{4} = \frac{19}{81},$$

$$C_{22} = \mathrm{Cov}(R_2, R_2) = \mathrm{Var}[R_2] = 1,$$

$$C_{12} = C_{21} = \mathrm{Cov}(R_1, R_2) = \mathbb{E}[R_1 R_2] = -\frac{7}{9} \cdot \frac{1}{4} - \frac{1}{9} \cdot \frac{1}{2} + \frac{5}{9} \cdot \frac{1}{4} = -\frac{1}{9}.$$

Hence the covariant matrix is

$$C = \begin{pmatrix} \frac{19}{81} & -\frac{1}{9} \\ -\frac{1}{9} & 1 \end{pmatrix},$$

whose inverse is

$$C^{-1} = \begin{pmatrix} \frac{9}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{19}{18} \end{pmatrix},$$

Replacing what we found in (B.9) we obtain,

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = -\begin{pmatrix} 9 & 1 \\ 1 & \frac{19}{9} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ -28/9 \end{pmatrix}.$$

Hence the number of shares in the portfolio are

$$\pi_1 = -10 \Rightarrow a_1 = -\frac{1000 \cdot 10}{36} \approx -278$$

$$\pi_2 = -28/9 \Rightarrow a_2 = -\frac{1000 \cdot 28}{2 \cdot 9} \approx -1556.$$

Hence the investor takes a short position on the risky assets and invest in the risk-free asset the capital $K = 1000$ plus the income from the short position. This is natural, because the risky assets have zero expected return in the present exercise. This concludes the second part of the exercise.

# Appendix C

# Answer to selected exercises

Exercise 2.2: $V^u(1) = 3, V^d(1) = 9$, $V(0) = 15$.

Exercise 2.3: $h_S(1) = -6, h_B(1) = 21$; if the price of the stock goes up at time $t = 1$, choose $h_S(2) = -1/3, h_B(2) = 4$; if the price of the stock goes down at time $t = 1$, choose $h_S(2) = -5/2$, $h_B(2) = 14$.

Exercise 3.4: $\Pi_Y(0) = 5/54$. $\mathbb{P}(Y > 0) = 28,125\%$.

Exercise 4.7: $\widehat{h}_S(1) = \widehat{h}_S(0) = -1/5$, $\widehat{h}_B(1) = \widehat{h}_B(0) = 14/45$; if the stock price goes up at time 1, then $\widehat{h}_S(2) = \widehat{h}_B(2) = 0$; if the price of the stock goes down at time 1, then $\widehat{h}_S(2) = -4/5, \widehat{h}_B(2) = 224/405$.

Exercise 5.6: $\mathbb{E}[X] = N(2p - 1)$.

Exercise 5.8: $\mathbb{E}[R] = -\frac{87}{112}$, $\mathrm{Var}[R] = 7983/12544 \approx 64\%$.

Exercise 6.11: $\Pi_Y(t) = S(t)\Phi(d_1)$.

Exercise 6.16: $\Pi_Y(0) = S_0 - S_0\Phi(d - \sigma\sqrt{T}) + B_0\Phi(d)$, where $d = \frac{1}{2}\sigma\sqrt{T} + \frac{1}{\sigma\sqrt{T}}\log\left(\frac{B_0}{S_0}\right)$. Moreover $\lim_{\sigma\to+\infty}\Pi_Y(0) = S_0 + B_0$ and $\lim_{\sigma\to 0^+}\Pi_Y(0) = \max(S_0, B_0)$.

Exercise A.3: For $i = 1, 2$ we have

$$\pi_i = \frac{1}{2\theta}\frac{e^{\mu_i T} - e^{rT}}{e^{2\mu_i T}(e^{\sigma_i^2 T} - 1)}.$$

# Appendix D

# Additional exercises

The exercises in this appendix are taken from [3] and from old exams.

## Chapter 1

For the following exercises assume that the dominance principle holds.

1. Let $r > 0$. Prove that $C(t, S(t), K, T) \to S(t)$ as $T \to \infty$
2. Prove that $\partial_K C(t, S(t), K, T) \geq -\exp(-r\tau)$, provided the derivative exists
3. Prove that $C(t, S(t), K, T) \to 0$ as $S(t) \to 0$
4. Prove that $\partial_s C(t, s, K, T) - \partial_s P(t, S(t), K, T) = 1$
5. Find a constant portfolio consisting of European calls and puts with expiration date $T$ such that the value of the portfolio at time $T$ equals $V(T) = \min(K, |S(T) - K|)$
6. Find a constant portfolio consisting of European calls and puts with expiration date $T$ such that the value of the portfolio at time $T$ equals $V(T) = \min[(S(T) - K)_+, (L - S(T))_+, (L - K)/4]$, where $0 < K < L$

## Chapter 3

In the following exercises, $\{S(t), B(t)\}_{t\in\mathcal{I}}$, $\mathcal{I} = \{1, \dots, N\}$, is an arbitrage free binomial market with parameters $d < r < u$, $p \in (0, 1)$. The value of the market parameters is specified in each single exercise.

1. Assume $d = -u$, $r = u/2$ and $N = 2$ and let $g(x) = 1$ if $x = 0$, $g(x) = 0$ if $x \neq 0$. Consider a European derivative with pay-off $Y = g(S(2) - S(0))$ at time

of maturity $T = N = 2$. Find $\Pi_Y(0)$ and $h_S(0)$. ANSWER: $\Pi_Y(0) = 2e^{-u}q_u q_d$, $h_S(0) = h_S(1) = \frac{e^{-u/2}}{S(0)}\frac{q_d - q_u}{e^u - e^{-u}}$

2. Let $u > r > 0$, $d = -u$ and $N = 3$. A European style derivative with expiration date $T = N = 3$ pays the amount 1 if the underlying stock moves always in the same direction (up or down) and 0 otherwise. Compute $\Pi_Y(0)$. ANSWER: $\Pi_Y(0) = e^{-3r}(q_u^3 + q_d^3)$

3. Let $N = 2$ and $e^r = (e^u + e^d)/2$. Derive the hedging portfolio of a European derivative paying the amount $Y = |S(2)/S(1) - S(1)/S(0)|$ at time of maturity $T = 2$

4. (Hard) A European derivative pays the amount $Y = \sum_{t=1}^{N}(S(t) - S(t-1))_+$ at maturity $T = N$. Find $\Pi_Y(0)$. ANSWER: $\Pi_Y(0) = S_0 A \frac{1-e^{-rN}}{e^r - 1}$, where $A = [q_u(e^u - 1)_+ + q_d(e^d - 1)_+]$

5. Let $N = 3$ and

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad S_0 = \frac{64}{25}, \quad p \in (0,1).$$

Consider a European derivative with maturity $T = 3$ and pay-off

$$Y = \Big(\frac{1}{4}\sum_{i=0}^{3} S(i) - 2\Big)_+,$$

which is an example of **Asian** call option. Compute the price of the derivative at time $t = 0$. Compute the probability that the derivative expires in the money and the probability that a long position in one share of the derivative gives a positive return. ANSWER: $\Pi_Y(0) = 52/75$, $\mathbb{P}(Y > 0) = p$, $\mathbb{P}(R > 0) = p^2$

# Chapter 4

In the following exercises, $\{S(t), B(t)\}_{t\in\mathcal{I}}$, $\mathcal{I} = \{1, \dots, N\}$, is an arbitrage free binomial market with parameters $d < r < u$, $p \in (0,1)$. The value of the market parameters is specified in each single exercise.

1. Let $N = 2$, $e^u = \frac{7}{4}$, $e^d = \frac{1}{2}$ and $e^r = \frac{9}{8}$. Consider an American put option with strike $K = 3/4$ at the maturity time $T = 2$. Assume $S(0) = B(0) = 1$. Compute the initial binomial price of the derivative and the initial position in the hedging portfolio. Verify if the put-call parity holds at all times. ANSWER: $\widehat{\Pi}_Y(0) = 1/9$, $h_S(0) = h_S(1) = -1/5$, $h_B(1) = h_B(0) = 14/45$

2. Assume $e^u = \frac{7}{4}$, $e^d = \frac{1}{2}$, $S(0) = 1$, $p = 3/4$, $e^r = 9/8$.

a) Compute the fair price at $t = 0, 1, 2$ of an American put with strike $K = 3/4$ and maturity $T = 2$

b) Compute the fair price at $t = 0, 1, 2$ of a European call with strike $K = 3/4$ and maturity $T = 2$

c) A derivative $\mathcal{U}$ gives to its owner the right to convert $\mathcal{U}$ at time $t = 1$ into either the European call or the American put defined above. Compute the fair price of $\mathcal{U}$ at time $t = 0$

d) Describe the *optimal* strategy that the holder of $\mathcal{U}$ should follow

e) Compute the expected value at time $t = 2$ of a portfolio containing one share of $\mathcal{U}$ at time $t = 0$ and assuming that the American put is not exercised at time $t = 1$

ANSWER: c) $\Pi_{\mathcal{U}}(0) = 16/27$, e) $\mathbb{E}[V(2)] = 347/256$

3. Consider the American option in Exercise 4.4. Assume either that (a) the buyer is not allowed to exercise at time $t = 1$ or that (b) the buyer is not allowed to exercise at time $t = 2$. In both cases compute the initial price of the option and discuss the difference with the original American option, where exercise is allowed at all times. REMARK: The new option where either (a) or (b) is enforced is an example of **bermuda** option.

# Chapter 5

1. Consider a portfolio that is long 1 share of the American put option in Section 4.3. Assume $p = 1/2$ and compute the expected relative return of the portfolio. ANSWER: $\mathbb{E}[R] = 17/648$

2. Repeat the previous exercise for the compound option in Exercise 3.2. ANSWER: $\mathbb{E}[R] = 77/16$

3. Suppose $X$, $Y$ are independent random variables with density functions $f_X$, $f_Y$ respectively. Prova that $X + Y$ has the density function

$$f_{X+Y}(x) = \int_{\mathbb{R}} f_X(y) f_Y(x - y)\, dy, \quad x \in \mathbb{R}.$$

4. Use the result in the previous exercise to prove Theorem 5.9.

# Chapter 6

In the following exercises, the stock price $S(t)$ is given by a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and instantaneous mean of log-return $\alpha$.

1. Find the Black-Scholes price of a European derivative with pay-off $Y = S(T) + S(T)^{-1}$ at maturity $T > 0$. ANSWER: $\Pi_Y(t) = S(t) + e^{(\sigma^2-2r)\tau} S(t)^{-1}$

2. Let $0 < a < b$. A European derivative pays 1 if $S(T)$ lies in the interval $(a, b)$ and zero otherwise. (a) Compute the Black-Scholes price $\Pi_Y(t)$ of the derivative and (b) find the value of $S(t)$ for which $\Pi_Y(0)$ is maximal. ANSWER (b): $S(0) = \sqrt{ab}e^{-(r-\frac{\sigma^2}{2})\tau}$

3. A European derivative pays 1 if $S(T) > K$ and $-1$ otherwise, where $K > 0$. Determine $K$ such that $\Pi_Y(0) = 0$. ANSWER: $K = S(0)e^{(r-\frac{\sigma^2}{2})T}$

4. Let $0 < L < K$. A European style derivative on a stock with maturity $T > 0$ pays nothing to its owner when $S(T) > K$, while for $S(T) < K$ it lets the owner choose between 1 share of the stock and the fixed amount $L$. (a) Draw the pay-off function of the derivative. (b) Compute the Black-Scholes price of the derivative. (c) Compute the number of shares of the stock in the hedging self-financing portfolio. ANSWER (c): $h_S(t) = \Delta(t, S(t))$, where $\Delta(t, x) = \Phi(d_2(K)) - \Phi(d_2(L)) - \phi(d_2(K))/\sigma\sqrt{\tau}$, where $d_2(a) = d_1(a) - \sigma\sqrt{\tau}$ and $d_1(a) = (\log \frac{a}{x} - (r - \frac{\sigma^2}{2})\tau)/\sigma\sqrt{\tau}$.

5. Consider a European derivative with pay-off $Y = (S(T) - S(0))^2/S(T)$ at time of maturity $T > 0$. Compute the Black-Scholes price $\Pi_Y(t)$ and the number of shares of the stock in the hedging portfolio. ANSWER: $\Pi_Y(t) = S(t) - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)r} S(t)^{-1}$; $h_S(t) = 1 - S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-2}$

6. Let $a, K, T > 0$ be given numbers and consider a European derivative with pay-off $Y = KH(a - S(T))$, where $H$ is the Heaviside function. (a) Compute the Black-Scholes price of the derivative; (b) compute the greeks of the derivative

7. Let $v(t, x)$ be the Black-Scholes price function of a European derivative with pay-off $Y = g(S(T))$. Show that $v$ is convex in the variable $x$; show also that $v(t, x) \geq g(x)$ if $g(0) = 0$. Finally prove that $v(t, x)$ satisfies the Black-Scholes PDE:

$$\partial_t v + rx\partial_x v + \frac{\sigma^2}{2} x^2 \partial_x^2 v - rv = 0, \quad 0 \leq t < T, \quad x > 0,$$

with terminal value $v(T, x) = g(x)$.

8. Let $C(t, x)$ be the Black-Scholes price function of a call option with strike $K$ and maturity $T$. Show that

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{x d_1 d_2}{\sigma} \phi(d_1)\sqrt{\tau}.$$

Conclude from this that the function $\sigma \to C(t, x)$ is convex in the interval $(0, \sigma_0]$ and concave in the interval $[\sigma_0, \infty)$, where

$$\sigma_0 = \sqrt{\frac{2}{\tau} |\log \frac{x e^{r\tau}}{K}|}$$

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