Application of the SOR Technique
to the
Solution of Electrostatic Problems by Finite Difference Methods

Consider the two dimensional rectangular region shown in the figure. The SOR technique requires that we execute the following equation recursively, starting from a guess:

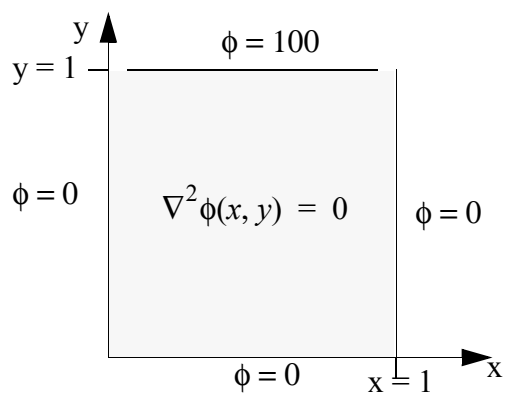
$$\phi_{i,j}^{(n+1)} = \omega\left[\frac{1}{4}(\phi_{i-1,j}^{(n)} + \phi_{i+1,j}^{(n)} + \phi_{i,j-1}^{(n)} + \phi_{i,j+1}^{(n)}) - \phi_{i,j}^{(n)}\right] + \phi_{i,j}^{(n)}$$

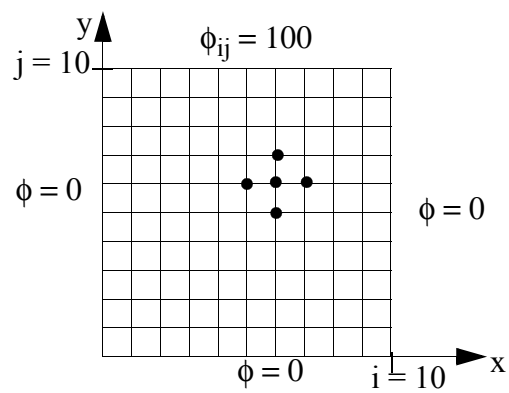
where ω is the acceleration factor. A pseudo code program to solve for the scalar potential $\phi(x,y)$ on a rectangular grid using Successive Over-Relaxation (SOR) is given below.

The relative displacement norm, ε, is used to stop the iterative loop. The relative displacement norm at the $(m+1)^{th}$ iteration is derived from the displacement norm δ as:

$$\delta = \sum_{i=1}^{N} \left|\phi_i^{(m+1)} - \phi_i^{(m)}\right| = \|\Delta\phi\| \qquad \varepsilon = \frac{\delta}{\|\phi\|} = \frac{\delta}{\sum_{i=1}^{N}\left|\phi_i^{(m+1)}\right|}$$

where N is the total number of grid points in the two dimensional grid. A grid with $h = \Delta x = \Delta y = 0.1$ is used.





The program checks the total flux emanating from the enclosed region by numerically integrating the normal component of the electric field around the boundary. This should be equal to zero. Why?

Pseudo Code - SOR Solution of Laplace's Equation on Rectangular Region

- Dirichlet conditions all around rectangular region
- Calculation of total flux emanating from region

```
declare  Φ(0-10, 0-10)                    solution matrix
input    ε, ω, max_iter                   relative displacement norm, acceleration factor, maximum iterations
for i = 0(1)10 repeat                     impose Dirichlet conditions
        set   Φ(i, 0) = 0.0, Φ(i, 10) = 100.0   bottom and top
              Φ(0, i) = 0.0, Φ(10, i) = 0.0     left and right sides
end
for i = 1(1)9 repeat                      initial guess, set all interior potential values to 1.0
        for j = 1(1)9 repeat
                set   Φ(i, j) = 1.0
        end
end
set   iter = 1, Err_norm = 99999                        count number of iterations
while ( iter < max_iter AND Err_norm > ε ) repeat       SOR iteration loop
        set ||Φ|| = 0, ||δ|| = 0                        solution norm, displacement norm
        for i = 1(1)9 repeat                            loop over inner matrix
                for j = 1(1)9 repeat
                        set Residual = ( Φ(i-1, j) + Φ(i+1, j) + Φ(i, j+1) + Φ(i, j-1) ) / 4.0
                            Acc_residual = ω * ( Residual - Φ(i, j) )
                            Φ(i, j) = Φ(i, j) + Acc_residual
                            ||Φ|| = ||Φ|| + | Φ(i, j) |         calculate current solution norm
                            ||δ|| = ||δ|| + | Acc_residual |    calculate displacement norm
                end
        end
        set Err_norm = ||δ|| / ||Φ||, iter = iter + 1           calculate relative displacement norm
end
set flux = 0                                            calculate total flux emanating from region
for i = 1(1)9 repeat
        set   flux = flux + Φ(0, i) + Φ(i, 0) + Φ(10, i) + Φ(i, 10)     outside potential
              flux = flux - Φ(1, i) - Φ(i, 1) - Φ(9, i) - Φ(i, 9)       inside potential
end
output   Φ, flux
```