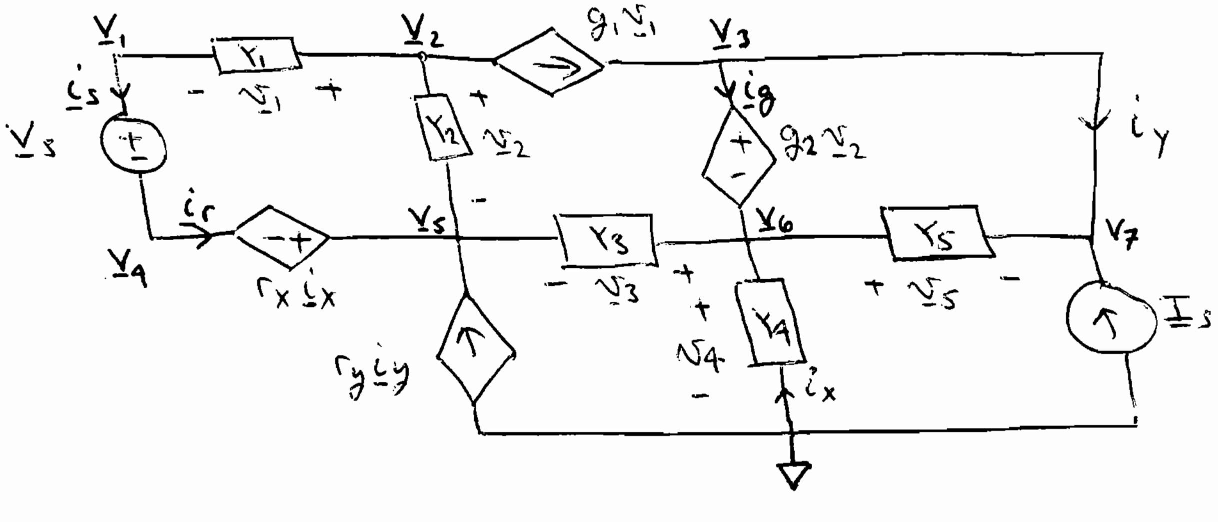
Modified Nodal Analysis (MNA)

In nodal analysis we use only nodal voltages to formulate enough equations to solve for the nodal voltages themselves.

This requires that we use "supernodes" around voltage sources and that we attempt to express controlling currents in terms of nodal voltages.

In modified nodal analysis (MNA) we keep controlling currents as variables to be solved for and we introduce auxiliary currents accross voltage sources (if that branch doesn't already contain a controlling current).

Example



We formulate enough equations to solve for the 7 nodal voltages and the 5 auxiliary currents (2 controlling currents: $\underline{i}_x, \underline{i}_y$ and 3 introduced auxiliary currents across voltage sources: i_s, i_r, i_g)

7 KCL Equations:

① $Y_1(\underline{V}_1 - \underline{V}_2) + \underline{i}_s = 0 \Rightarrow Y_1\underline{V}_1 - Y_2\underline{V}_2 + \underline{i}_s = 0$

② $Y_1(\underline{V}_2 - \underline{V}_1) + Y_2(\underline{V}_2 - \underline{V}_5) + g_1(\underline{V}_2 - \underline{V}_1) = 0$

$\Rightarrow -(Y_1 + g_1)\underline{V}_1 + (Y_1 + Y_2 + g_1)\underline{V}_2 - Y_2\underline{V}_5 = 0$

③ $-g_1(\underline{V}_2 - \underline{V}_1) + i_y + i_g = 0$

$\Rightarrow g_1\underline{V}_1 - g_1\underline{V}_2 + i_y + i_g = 0$

④ $-i_s + i_r = 0$

⑤ $Y_2(\underline{V}_5 - \underline{V}_2) + Y_3(\underline{V}_5 - \underline{V}_6) - r_y i_y - i_r = 0$

$\Rightarrow -Y_2\underline{V}_2 + (Y_2 + Y_3)\underline{V}_5 - Y_3\underline{V}_6 - r_y i_y - i_r = 0$

⑥ $Y_3(\underline{V}_6 - \underline{V}_5) + Y_5(\underline{V}_6 - \underline{V}_7) - i_x - i_g = 0$

$\Rightarrow -Y_3\underline{V}_5 + (Y_3 + Y_5)\underline{V}_6 - Y_5\underline{V}_7 - i_x - i_g = 0$

⑦ $-Y_5\underline{V}_6 + Y_5\underline{V}_7 - i_y = \underline{I}_s$

5 Aux. Eq:

⑧ $\underline{V}_1 - \underline{V}_4 = \underline{V}_s$

⑨ $-\underline{V}_4 + \underline{V}_5 - r_x i_x = 0$

⑩ $\underline{V}_3 - \underline{V}_6 = g_2(\underline{V}_2 - \underline{V}_5)$

$\Rightarrow -g_2\underline{V}_2 + \underline{V}_3 + g_2\underline{V}_5 - \underline{V}_6 = 0$

⑪ $\underline{V}_3 - \underline{V}_7 = 0$ (short circuit is equivalent to zero voltage voltage source)

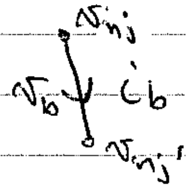
⑫ $Y_4\underline{V}_6 + i_x = 0$

The matrix equation can now be formulated and solved on the computer.

MNA by Inspection (Using "stamps")

- the node voltages will be the initial variable that we want to solve for. (n-variables) We know we can write n KCL equations.
- so when we write KCL at a node, all elements connected to that node which have an admittance description,

$$Y_b v_b = i_b$$



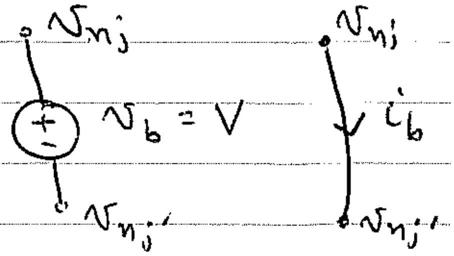
the current i_b involved in KCL can be written in terms of the node voltages:

$$Y_b(v_{nj} - v_{nj'}) = i_b \quad \text{KCL at node } j$$
$$Y_b(v_{nj'} - v_{nj}) = -i_b \quad \text{KCL at node } j'$$

$\therefore$ in the matrix we would introduce the "stamp":

$$\begin{array}{c|cc} & v_{nj} & v_{nj'} \\ \hline j & Y_b & -Y_b \\ j' & -Y_b & Y_b \end{array}$$

- for a voltage soure when it is involved in KCL we need to use the branch current i_b, $\therefore$ i_b will be added as a variable



but if we've introduce a new variable to solve for, then we need another equation. That equation is the constitutive relation for the voltage source:

$$v_{nj} - v_{nj'} = V$$

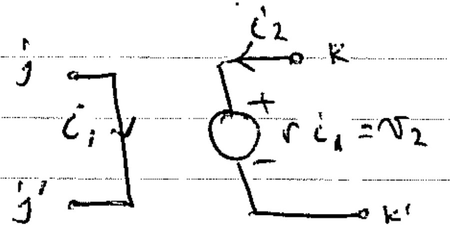
So the stamp becomes:

$$\begin{array}{l|cc:c|c|}
 & v_{nj} & v_{nj'} & i_b & \text{RHS (source vector)} \\
\text{KCL@ } j & & & 1 & \\
\text{KCL@ } j' & & & -1 & \\
\hdashline
\text{constitutive relation} & 1 & -1 & & V
\end{array}$$

The pattern for the stamps is always the same

$$\begin{array}{l|cc:cc|}
 & v_{nj} & v_{nj'} & \text{aux. currents } i_{b1} & i_{b2} \\
\text{KCL@ } j & & & & \\
\text{KCL@ } j' & & & & \\
\text{Const. relation} & & & & \\
 & & & &
\end{array}$$

Example: CVT
(2-port)



$v_{nj} - v_{nj'} = 0$

$v_{nk} - v_{nk'} - r i_1 = 0$

KCL@ j : $I_j = i_1$

KCL@ j' : $I_{j'} = -i_1$

KCL@ k : $I_K = i_2$

KCL@ k' : $I_k = -i_2$

$$\left[\begin{array}{cccc:c:c}
 & & & & i_1 & i_2 \\
 & & & & 1 & \\
 & & & & -1 & \\
 & & & & & 1 \\
 & & & & & -1 \\
\hdashline
1 & -1 & & & & \\
\hdashline
 & & 1 & -1 & -r &
\end{array}\right]$$

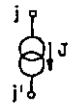
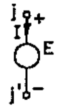
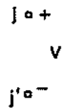
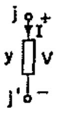
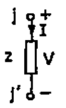
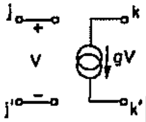
ELEMENT	SYMBOL	MATRIX	EQUATIONS
CURRENT SOURCE		$\begin{array}{cc} j & -J \\ j' & J \end{array}$ SOURCE VECTOR	$I_j = J$ $I_{j'} = -J$
VOLTAGE SOURCE		$\begin{array}{cccc} & V_j & V_{j'} & I \\ j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & \end{array}$ SOURCE VECTOR $\begin{array}{c} \\ \\ E \end{array}$	$V_j - V_{j'} = E$ $I_j = I$ $I_{j'} = -I$
OPEN CIRCUIT		———	$V = V_j - V_{j'}$
SHORT CIRCUIT		$\begin{array}{cccc} & V_j & V_{j'} & I \\ j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & \end{array}$	$V_j - V_{j'} = 0$ $I_j = I$ $I_{j'} = -I$
ADMITTANCE		$\begin{array}{ccc} & V_j & V_{j'} \\ j & y & -y \\ j' & -y & y \end{array}$	$I_j = y(V_j - V_{j'})$ $I_{j'} = -y(V_j - V_{j'})$
IMPEDANCE		$\begin{array}{cccc} & V_j & V_{j'} & I \\ j & & & 1 \\ j' & & & -1 \\ m+1 & 1 & -1 & -z \end{array}$	$V_j - V_{j'} - zI = 0$ $I_j = -I_j' = I$
NULLATOR		$\begin{array}{ccc} & V_j & V_{j'} \\ & & \\ m+1 & 1 & -1 \end{array}$	$V_j - V_{j'} = 0$ $I_j = I_{j'} = 0$
NORATOR		$\begin{array}{ccc} & & I \\ j & & 1 \\ j' & & -1 \end{array}$	V, I ARE ARBITRARY
VCT		$\begin{array}{ccc} & V_j & V_{j'} \\ k & g & -g \\ k' & -g & g \end{array}$	$I_j = 0$ $I_j' = 0$ $I_k = g(V_j - V_{j'})$ $I_{k'} = -g(V_j - V_{j'})$

Fig. 4.4.1. Ideal elements in the modified nodal formulation without graphs.

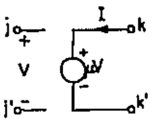
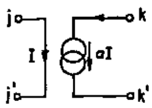
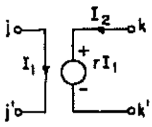
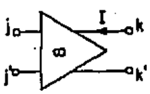
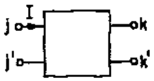
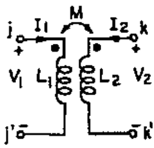
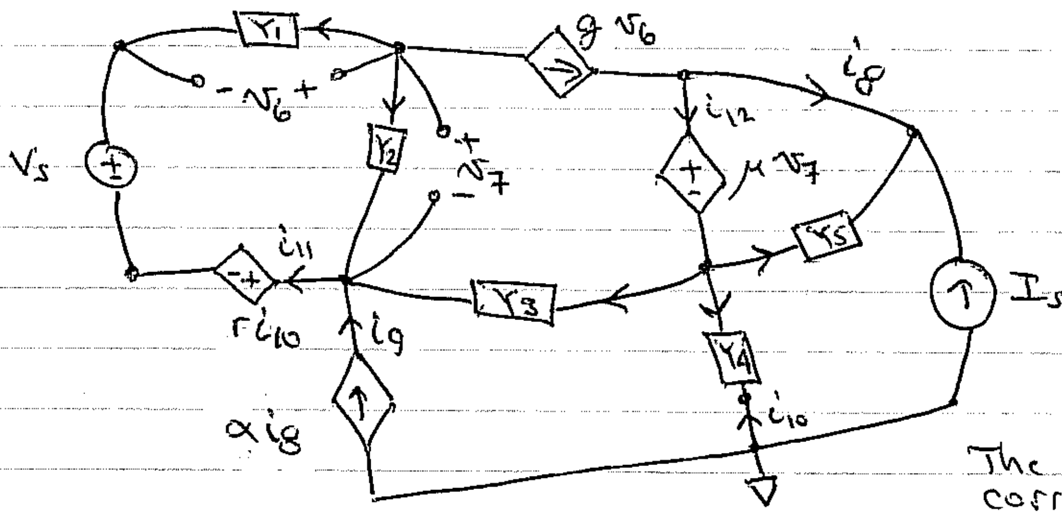
ELEMENT	SYMBOL	MATRIX	EQUATIONS
VVT		$\begin{array}{cccccc} & V_j & V_{j'} & V_k & V_{k'} & I \\ j & & & & & \\ j' & & & & & \\ k & & & & & 1 \\ k' & & & & & -1 \\ m+1 & -\mu & \mu & 1 & -1 & \end{array}$	$-\mu V_j + \mu V_{j'} + V_k$ $-V_{k'} = 0$ $I_k = I$ $I_{k'} = -I$
CCT		$\begin{array}{cccccc} & V_j & V_{j'} & V_k & V_{k'} & I \\ j & & & & & 1 \\ j' & & & & & -1 \\ k & & & & & \alpha \\ k' & & & & & -\alpha \\ m+1 & 1 & -1 & & & \end{array}$	$V_j - V_{j'} = 0$ $I_j = -I_{j'} = I$ $I_k = -I_{k'} = \alpha I$
CVT		$\begin{array}{ccccccc} & V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \\ j & & & & & 1 & \\ j' & & & & & -1 & \\ k & & & & & & 1 \\ k' & & & & & & -1 \\ m+1 & 1 & -1 & & & & \\ m+2 & & & 1 & -1 & -r & \end{array}$	$V_j - V_{j'} = 0$ $V_k - V_{k'} - rI_1 = 0$ $I_j = -I_{j'} = I_1$ $I_k = -I_{k'} = I_2$
OPERATIONAL AMPLIFIER		$\begin{array}{cccccc} & V_j & V_{j'} & V_k & V_{k'} & I \\ j & & & & & \\ j' & & & & & \\ k & & & & & 1 \\ k' & & & & & -1 \\ m+1 & 1 & -1 & & & \end{array}$	$V_j - V_{j'} = 0$ $I_k = -I_{k'} = I$
CONVERTOR		$\begin{array}{cccccc} & V_j & V_{j'} & V_k & V_{k'} & I \\ j & & & & & 1 \\ j' & & & & & -1 \\ k & & & & & -K_2 \\ k' & & & & & K_2 \\ m+1 & 1 & -1 & -K_1 & K_1 & \end{array}$	$V_j - V_{j'} - K_1 V_k + K_1 V_{k'} = 0$ $I_j = -I_{j'} = I$ $I_k = -I_{k'} = -K_2 I$ FOR IDEAL TRANSFORMER $K_1 = K_2 = n$
TRANSFORMER		$\begin{array}{ccccccc} & V_j & V_{j'} & V_k & V_{k'} & I_1 & I_2 \\ j & & & & & 1 & \\ j' & & & & & -1 & \\ k & & & & & & 1 \\ k' & & & & & & -1 \\ m+1 & 1 & -1 & & & -sL_1 & -sM \\ m+2 & & & 1 & -1 & -sM & -sL_2 \end{array}$	$V_j - V_{j'} - sL_1 I_1 - sM I_2 = 0$ $V_k - V_{k'} - sM I_1 - sL_2 I_2 = 0$ $I_j = -I_{j'} = I_1$ $I_k = -I_{k'} = I_2$

Fig. 4.4.1. *(Continued)*

Let's redo the previous example using the stamps in Fig. 4.4.1.

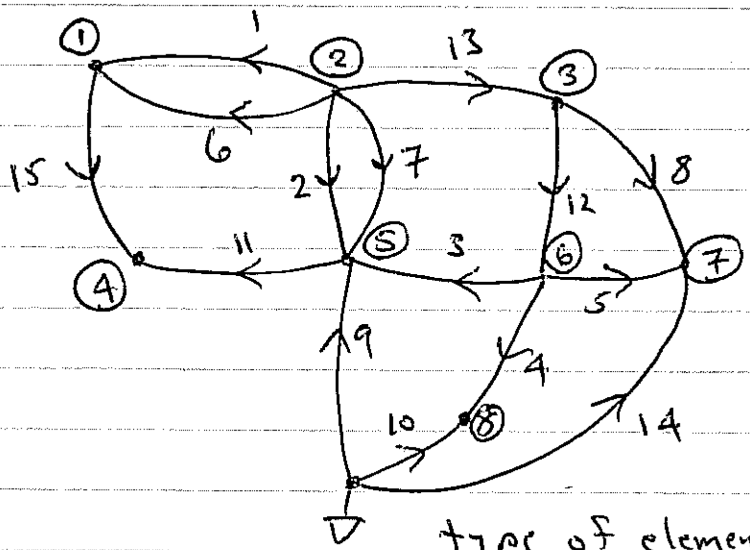
First we should redraw the network so that all branches for the two-ports are clearly identifiable.



Note that i_{10} was added a one side of CVT

v_6 as one side of VCT and v_7 as one side of VVT

The CCT was already correctly represented.



In MNA, it is the nodal voltages which are solved for. So there was no need to number the branches.

The information can be written as a list of elements, the type of element, the nodes at the terminals 'j,j'' for one ports, 'j, j', k, k' for two ports, and the appropriate parameters which define the constitutive relations for that element.

Let's assume that branches 1 and 2 are given as admittances G_1 and sC_2 while branches 3, 4, and 5 are impedances R_3, R_4, sL_5.

Type	Nodes	Parameters	
G	2 1	G_1	
C	2 5	C_2	entered as sC_2 or $j\omega C_2$
R	6 5	R_3	adds i_{65}
R	6 8	R_4	adds i_{68}
L	6 7	L_5	adds i_{67} and entered as $-sL_5$ or $-j\omega L_5$
VCT	2 1 2 3	g	no aux. current
VVT	2 5 3 6	μ	aux. current i_{36}
CCT	3 7 0 5	α	aux. currents i_{37}
CVT	0 8 5 4	r	aux. currents i_{08}, i_{54}
CS	0 7	I_s	RHS
VS	1 4	V_s	aux. current i_{14} and RHS

We will have 8 nodal variables and 8 auxiliary variables in our MNA equation.

The CVT adds two auxiliary currents
The CCT and VVT both add one
The VS adds one

Each of the impedance variables adds one auxiliary current

The Final metrix equation will look like $T\underline{x} = \underline{w}$

T is 16×16 $(T = G + sC)$ $\quad \underline{x} = \begin{pmatrix} \underline{v}_n \\ \underline{i}_a \end{pmatrix}$ $\quad \underline{v}_n$ is 8×1 $\quad \underline{w} = 16 \times 1$

$\underline{i}_a$ is 8×1

For the auxiliary currents we can name them as per the nodes they fall between:

$$\underline{i}_a = \begin{array}{c} \left(\begin{array}{c} i_{65} \\ i_{68} \\ i_{67} \\ i_{36} \\ i_{37} \\ i_{08} \\ i_{54} \\ i_{14} \end{array}\right) \end{array} \begin{array}{c} 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array} \qquad \underline{v} = \begin{pmatrix} v_{n1} \\ v_{n2} \\ v_{n3} \\ v_{n4} \\ v_{n5} \\ v_{n6} \\ v_{n7} \\ v_{n8} \end{pmatrix}$$

$$G \; 2 \; 1 \; G_1 \qquad \begin{array}{c|cc} & 2 & 1 \\ \hline 2 & G_1 & -G_1 \\ 1 & -G_1 & G_1 \end{array}$$

$$C \; 2 \; 5 \; C_2 \qquad \begin{array}{c|cc} & 2 & 5 \\ \hline 2 & sC_2 & -sC_2 \\ 5 & -sC_2 & sC_2 \end{array}$$

$$R \; 6 \; 5 \; R_3 \qquad \begin{array}{c|cc|c} & 6 & 5 & 9 \\ \hline 6 & & & 1 \\ 5 & & & -1 \\ \hline 9 & 1 & -1 & -R_3 \end{array}$$

$$R \; 6 \; 8 \; R_4 \qquad \begin{array}{c|cc|c} & 6 & 8 & 10 \\ \hline 6 & & & 1 \\ 8 & & & -1 \\ \hline & 1 & -1 & -R_4 \end{array}$$

L 6 7 Ls

$$\begin{array}{c|cc:c|} 6 & & & 1 \\ 7 & & & -1 \\ \hdashline 11 & 1 & -1 & -sL_s \end{array}$$

VCT 2123 g

$$\begin{array}{c|cc|} & 2 & 1 \\ 2 & g & -g \\ 3 & -g & g \end{array}$$

← note "2 1"

$$\Downarrow$$

$$\begin{array}{c|cc|} & 1 & 2 \\ 2 & -g & +g \\ 3 & +g & -g \end{array}$$

VVT 2536 μ

$$\begin{array}{c|cccc:c|} & 2 & 5 & 3 & 6 & \\ 2 & & & & & \\ 5 & & & & & \\ 3 & & & & & 1 \\ 6 & & & & & -1 \\ \hdashline 12 & -\mu & \mu & 1 & -1 & \end{array}$$

CCT 3705 α

$$\begin{array}{c|ccc:c|} & 3 & 7 & 5 & \\ 3 & & & & 1 \\ 7 & & & & -1 \\ 5 & & & & -\alpha \\ \hdashline 13 & 1 & -1 & & \end{array}$$

CVT 0854 r

$$\begin{array}{c|ccc:c:c|} & 8 & 5 & 4 & & \\ 8 & & & & -1 & \\ 5 & & & & & 1 \\ 4 & & & & & -1 \\ \hdashline 14 & -1 & & & & \\ \hdashline 15 & & 1 & -1 & 1 & -r \end{array}$$

CS 0 7 Is $\quad w \Rightarrow \quad w_7 = I_s$

VS 1 4 Vs

$$\begin{array}{c|cc:c|} & 1 & 4 & \\ 4 & & & -1 \\ \hdashline 16 & 1 & -1 & 1 \end{array} \quad w_{16} = V_s$$