Chapter 4: General Formulation Methods.

We will look at two methods:

1) Tableau Formulation
2) Modified Nodal Analysis. (MNA)

Finally, for Lab I you will implement an MNA parser which will automatically generate the MNA matrices from a Spice-like network specification.

Tableau Formulation

- "everything" is included here: KCL, KVL and the constitutive relations, the variables are branch currents, branch voltages, and nodal voltages.

- Because the incidence matrix A is easier to formulate, it is used.

KCL: $$\boxed{A \underline{i}_b = 0}$$ $\left\{\begin{array}{l} n \text{ - equations} \\ b \text{ - unknowns. } \underline{i}_b \end{array}\right.$

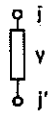
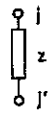
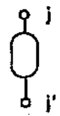
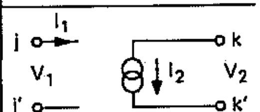
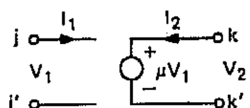
KVL: $$\boxed{\underline{v}_b - A^T \underline{v}_n = 0}$$ $\left\{\begin{array}{l} b \text{ - equations} \\ b \text{ - unknowns } \underline{v}_b \\ n \text{ - unknowns } \underline{v}_n \end{array}\right.$

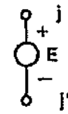
Constitutive Relations:

$$\begin{pmatrix} Y_1 \\ K_2 \end{pmatrix} \underline{v}_b + \begin{pmatrix} K_1 \\ Z_2 \end{pmatrix} \underline{i}_b = \begin{pmatrix} \underline{w}_{b1} \\ \underline{w}_{b2} \end{pmatrix}$$

or

$$\boxed{Y_b \underline{v}_b + Z_b \underline{i}_b = \underline{W}_b}$$ $\{ b \text{ - equations}$

ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
OPEN CIRCUIT		$I = 0$
SHORT CIRCUIT		$V = 0$
ADMITTANCE		$yV - I = 0$
IMPEDANCE		$V - zI = 0$
NULLATOR		$I = 0$ $V = 0$
NORATOR		I, V ARBITRARY (NO CONSTITUTIVE EQUATIONS)
VCT	 $I_1 = 0;\ I_2 = gV_1$	$\begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
VVT	 $I_1 = 0;\ V_2 = \mu V_1$	$\begin{bmatrix} 0 & 0 \\ \mu & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
CURRENT SOURCE		$I = J$
VOLTAGE SOURCE		$V = E$

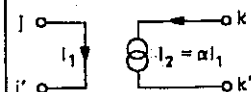
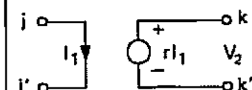
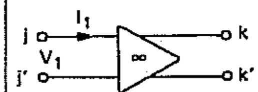
ELEMENT	SYMBOL	CONSTITUTIVE EQUATIONS
CCT	 $V_1 = 0;\ I_2 = \alpha I_1$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
CVT	 $V_1 = 0;\ V_2 = rI_1$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
OPAMP	 $V_1 = 0;\ I_1 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Table 4.1.3 from Vlach & Singhal

Some constitutive relations:

Resistor: $v_b - R_b i_b = 0$

Conductor: $G_b v_b - i_b = 0$

Capacitor: $sC_b v_b - i_b = C_b V_0$ V_0 - initial voltage.

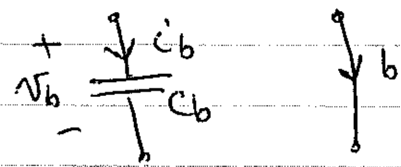
Inductor: $v_b - sL_b i_b = -L_b I_0$ I_0 - initial current.

ind. voltage source: $v_b = E_b$.

ind. current source: $i_b = J_b$

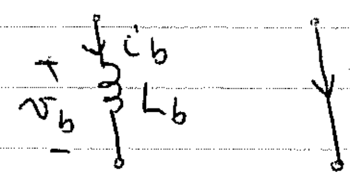
Note: we're using the Laplace transform to make the formulation more concise.

e.g.: $i_b = C_b \dfrac{dv_b}{dt}$



L.T.: $i_b = sC_b v_b - C_b V_0$ V_0 = initial voltage

e.g.: $v_b = L_b \dfrac{di_b}{dt}$



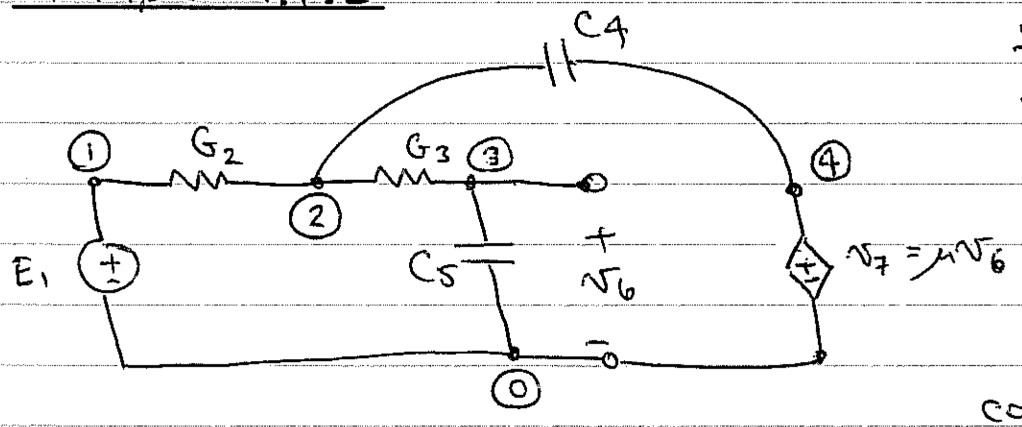
L.T.: $v_b = sL_b i_b - L_b I_0$ I_0 = initial current

Putting these all into one matrix:

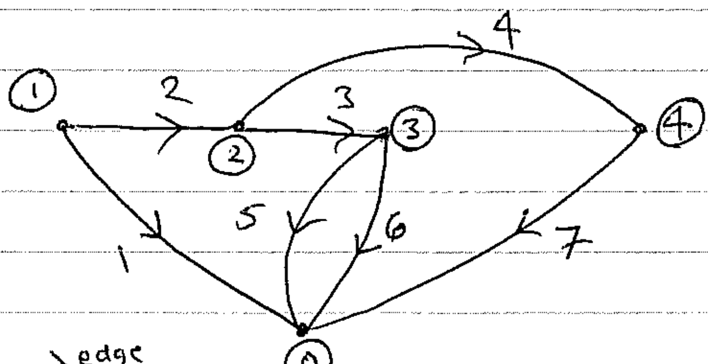
$$
\begin{array}{l}
b \text{ KVL} \\
b \text{ const.} \\
n \text{ KCL}
\end{array}
\begin{bmatrix} \mathbb{1} & 0 & -A^T \\ Y_b & Z_b & 0 \\ 0 & A & 0 \end{bmatrix}
\begin{pmatrix} \underline{v}_b \\ \underline{i}_b \\ \underline{v}_n \end{pmatrix}
=
\begin{pmatrix} 0 \\ \underline{w}_b \\ 0 \end{pmatrix}
$$

$$\boxed{T \underline{x} = \underline{W}}$$

Example 4.1.2



this circuit has a VVT two-port (voltage-to-voltage transducer).



constitutive relations:

$v_1 = E_1$

$G_2 v_2 - i_2 = 0$

$G_3 v_3 - i_3 = 0$

$s C_4 v_4 - i_4 = 0$

$s C_5 v_5 - i_5 = 0$

$$\text{VVT} \begin{cases} i_6 = 0 \\ \mu v_6 - v_7 = 0 \end{cases}$$

$$
A = \begin{array}{c|ccccccc}
\text{node} \backslash \text{edge} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\
4 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{array}
$$

The constitutive relations can be written as

$$Y_b v_b + Z_b i_b = \underline{w}_b$$

$$Y_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & sC_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sC_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & -1 \end{bmatrix}$$

$$Z_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{w}_b = \begin{pmatrix} E_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can reduce the number of equations by block elimination:

$$\begin{cases} \text{①} & \underline{v}_b = A^T \underline{v}_n \\ \text{②} & Y_b \underline{v}_b + Z_b \underline{i}_b = \underline{w}_b \\ \text{③} & A \underline{i}_b = 0 \end{cases}$$

① into ② : $Y_b A^T v_n + Z_b \underline{i}_b = \underline{w}_b$

③ : $A i_b = 0$

$$\underbrace{\begin{bmatrix} Y_b A^T & Z_b \\ 0 & A \end{bmatrix}}_{(n+b)\times(n+b)} \begin{pmatrix} v_n \\ i_b \end{pmatrix} = \begin{pmatrix} \underline{w}_b \\ 0 \end{pmatrix}$$

If every element has an admittance representation, and only current sources:

new ② $\underline{i}_b = Y_b \underline{v}_b + \underline{J}_b$ (not available for all elements, e.g. volt. src., VVT, CVT, CCT etc.).

using ① $\longrightarrow$ $\underline{i}_b = Y_b A^T \underline{v}_n + \underline{J}_b$

into ③ $\boxed{A Y_b A^T \underline{v}_n = - A \underline{J}_b}$ Nodal Analysis.

We need to add "auxiliary currents" into the nodal analysis formulation in order to handle all types of elements. This brings us to Modified Nodal Analysis. (MNA).

MNA USING FORMAL METHODS

In order to eliminate branch currents having an admittance description we must group these branches separately.

Let:

1) $\underline{i}_1$ be a vector of branch currents of elements with an admittance description.

2) $\underline{i}_2$ be a vector of branch currents of elements without an admittance description
AND
branch currents of voltage sources
AND
branch currents required as solution

3) $\underline{j}$ contains independent current sources.

Then the node-edge incidence matrix can be partition accordingly as:

$$\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \underline{i}_1 \\ \underline{i}_2 \\ \underline{j} \end{bmatrix} = 0 \qquad (KCL)$$

KVL: $\begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_j \end{pmatrix} = \begin{pmatrix} A_1^T \\ A_2^T \\ A_3^T \end{pmatrix} \underline{v}_n$

$$\underline{v}_1 = A_1^T \underline{v}_n$$

$$\underline{v}_2 = A_2^T \underline{v}_n$$

$$\underline{v}_j = A_3^T \underline{v}_n$$ (voltages across current sources in terms of $\underline{v}_n$)

In partition 2, the constitutive relation for the elements are written as:

$$Y_2 \underline{v}_2 + Z_2 \underline{i}_2 = \underline{w}_2$$ (non-zero only for voltage sources, current srcs are in $\underline{j}$ partition).

In partition 1, we have:

$$Y_1 \underline{v}_1 = \underline{i}_1$$

Re-write KCL:

$$A_1 \underline{i}_1 + A_2 \underline{i}_2 = -A_3 \underline{j}$$

$$A_1 Y_1 \underline{v}_1 + A_2 \underline{i}_2 = -A_3 \underline{j}$$

in terms of nodal voltages: $\boxed{A_1 Y_1 A_1^T \underline{v}_n + A_2 \underline{i}_2 = -A_3 \underline{j}}$

We can also substitute partition 2 (KVL) into the constitutive relations:

$$\boxed{Y_2 A_2^T \underline{v}_n + Z_2 \underline{i}_2 = \underline{w}_2}$$

Thus we have:

$$\begin{bmatrix} A_1 Y_1 A_1^T & A_2 \\ Y_2 A_2^T & Z_2 \end{bmatrix} \begin{pmatrix} \underline{v}_n \\ \underline{i}_2 \end{pmatrix} = \begin{pmatrix} -A_3 \underline{j} \\ \underline{w}_2 \end{pmatrix}$$

Let $\quad Y_{N1} = A_1 Y_1 A_1^T$

$$-A_3 \underline{\hat{j}} = \underline{j}_n$$

$$\begin{array}{l} \\ \text{KCL} \\ \text{additional equations} \end{array} \begin{array}{c} \begin{array}{cc} \text{node voltages} & \text{aux. currents} \end{array} \\ \begin{bmatrix} Y_{N1} & A_2 \\ Y_2 A_2^T & Z_2 \end{bmatrix} \end{array} \begin{pmatrix} \underline{v}_n \\ \underline{i}_2 \end{pmatrix} = \begin{pmatrix} \underline{j}_n \\ \underline{w}_2 \end{pmatrix} \begin{array}{l} \text{current sources @ nodes} \\ \text{voltage sources} \end{array}$$

Although this is a nice formal method to construct the MNA equations, it is not used in practice.

Instead, we use "stamps" and run thru the elements one at a time, introducing the corresponding stamp into the MNA matrices.