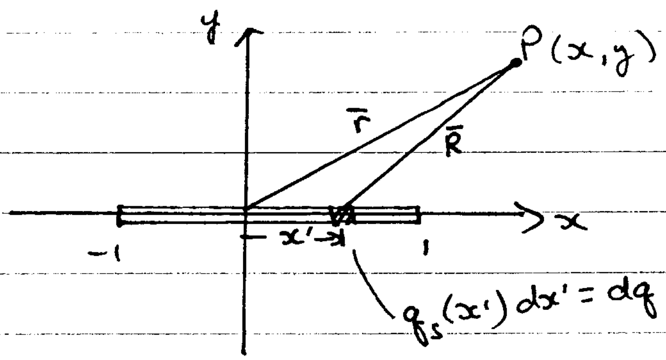
Integral Equations Solved by the Method of Moments – Example.

Consider the two dimensional problem of the infinite length charged strip:



observation point

$q_s(x')$ is the surface charge density per unit length of strip.

The potential at the point P due to an elemental charge dq located at $(x',0)$ is given by

$$d\Phi = \frac{dq}{2\pi\epsilon_0}\ln\frac{1}{|\bar{R}|} \qquad \bar{R} = \bar{r} - x'\hat{a}_x$$

(Green's fcn)

$\therefore$ total potential can be obtained by integrating over source location:

$$\phi(\bar{r}) = \frac{1}{2\pi\epsilon_0}\int_{-1}^{+1} q_s(x')\ln\frac{1}{|\bar{r}-x'\hat{a}_x|}\,dx' \qquad (1)$$

If we are not given $q_s(x')$ but rather the voltage on the strip, say V_0,

then $$V_0 = \frac{1}{2\pi\epsilon_0}\int_{-1}^{1} q_s(x')\ln\frac{1}{|x-x'|}\,dx' \qquad (2)$$

in ② x is any location on the strip since the voltage is constant on the whole strip.

let $F(x) = \dfrac{q_s(x)}{2\pi \epsilon_0 V_0}$

$\therefore$ we have an integral equation:

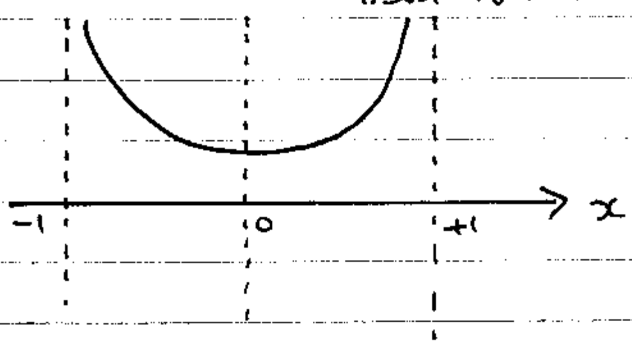
$$1 = \int_{-1}^{1} F(x') \, K(x, x') \, dx'$$

$K(x, x') = \ln \dfrac{1}{|x - x'|}$ is called the Kernel of the integral equation

the exact solution is found by using the identity

$$\int_{-1}^{+1} \frac{1}{\sqrt{1 - x'^2}} \ln \frac{1}{|x - x'|} \, dx' = \pi \ln 2$$

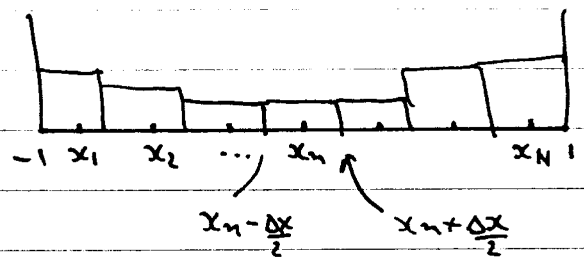
$$\therefore \quad F(x) = \frac{1}{\pi \ln 2 \sqrt{1 - x'^2}}$$



singularities at ± 1
$\Rightarrow$ ∞ charge densi

Numerical Solution using MOM

let us try expanding $f(x)$ as a superposition of pulse fcns:



$\Delta x = \dfrac{2}{N}$ N - total # of intervals

centre of each point: $x_n = -1 + (n-\tfrac{1}{2})\Delta x$
$n = 1(1)N$

pulse Function: $P_n(x) = \begin{cases} 1 & |(x-x_n)| \le \frac{\Delta x}{2} \\ 0 & \text{otherwise} \end{cases}$

$$f(x) = \sum_{n=1}^{N} \alpha_n P_n(x) \qquad \text{- expansion}$$

substitution into integral equation:

$$1 = \int_{-1}^{+1} f(x')\, K(x,x')\, dx' = \sum_{n=1}^{N} \alpha_n \int_{-1}^{+1} P_n(x')\, K(x,x')\, dx'$$

$$= \sum_{n=1}^{N} \alpha_n \int_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}} K(x,x')\, dx'$$

if we use collocation (i.e. testing fcns are delta fcns) then

$$\sum_{n=1}^{N} \alpha_n \int_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}} K(x_m, x')\,dx' = 1 \qquad m = 1(1)N$$

$$\sum_{n=1}^{N} \alpha_n K_{mn} = 1 \qquad m = 1(1)N$$

where:

$$K_{mn} = \int_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}} K(x_m, x')\,dx' = \int_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}} \ln \frac{1}{|x_m - x'|}\,dx'$$

but

$$\int \ln \frac{1}{(ax+b)}\,dx = -\int \ln(ax+b)\,dx$$

$$= -\frac{(ax+b)}{a} \ln(ax+b) + x$$

if $m > n \Rightarrow x_m > x'$

$$K_{mn} = \int_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}} \ln \frac{1}{x_m - x'}\,dx'$$

$$= \left[+ \frac{(-x' + x_m)}{+1} \ln(x_m - x') + x' \right]_{x_n-\frac{\Delta x}{2}}^{x_n+\frac{\Delta x}{2}}$$

$\vdots$

straight forward algebra

Method of Moments for Electrostatic Integral Equations

Poisson's equation: $\nabla^2 \phi = -\dfrac{\rho}{\epsilon}$

$$\bar{E} = -\nabla \phi$$

in unbounded space we require that

B.C. $r\phi \Rightarrow \text{constant as } r \Rightarrow \infty$

where r is distance from the origin and the charge density is of finite extent.

if we rewrite Poisson's equation as

$$-\epsilon \nabla^2 \phi = \rho$$

then $L\phi = \rho$ where $L = -\epsilon \nabla^2$

Domain of L is those fcns ϕ whose Laplacian exists and satisfy the B.C.'s.

The general solution is

$$\phi(x,y,z) = \iiint \frac{\rho(x',y',z')}{4\pi\epsilon R}\, dx'\, dy'\, dz'$$

where $$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

is the distance from the source location (x', y', z') to the field point (x, y, z)

Thus we have the inverse operator:

$$L^{-1} = \iiint dx'\,dy'\,dz' \frac{1}{4\pi\epsilon R}$$

a suitable inner product is

$$(\phi, \psi) = \iiint_{\substack{\text{all}\\ \text{space}}} \phi(x,y,z)\,\psi(x,y,z)\,dx\,dy\,dz.$$

This problem can be shown to be self-adjoint and positive-definite.

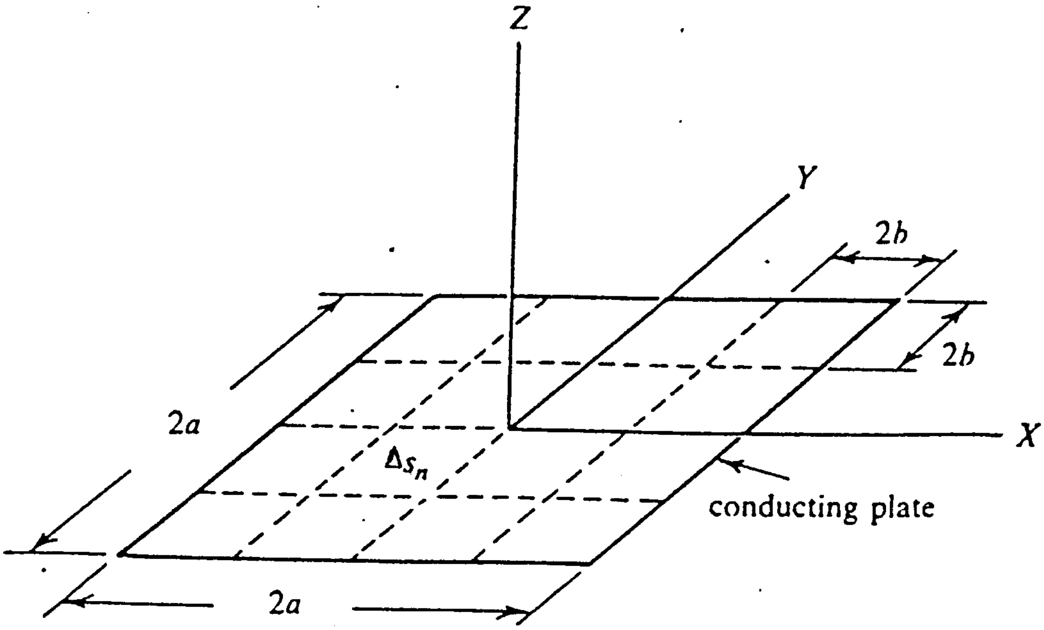


Figure 2-1. **Square conducting plate and subsections.**

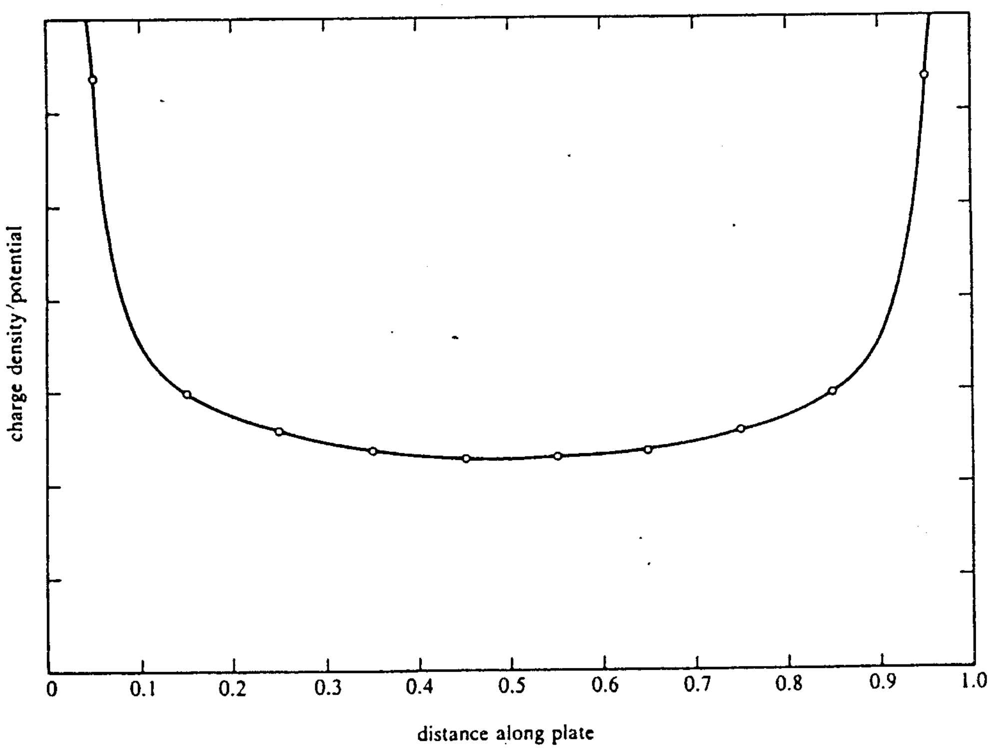


Figure 2-2. **Approximate charge density on subsections adjacent to the centerline of a square conducting plate.**

Charged Conducting Plate

Consider square conducting plate 2a meters on a side lying on the $z=0$ plane with center at the origin. (see Figure 2.1).

Then the electrostatic potential can be derived at any point in space if we know the surface charge density on the plate, $\sigma(x',y')$ as

$$\begin{cases} \phi(x,y,z) = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \, \dfrac{\sigma(x',y')}{4\pi\epsilon R} & ① \\ R = \sqrt{(x-x')^2+(y-y')^2+z^2} & ② \end{cases}$$

But say we are not given $\sigma(x',y')$ but only that the potential on the plate is constant an $\phi = V$ on the plate.

Then writing ① for any point on the plate $|x| < a$, $|y| < a$, $z=0$

$$V = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \, \frac{\sigma(x',y')}{4\pi\epsilon\sqrt{(x-x')^2+(y-y')^2}} \qquad ③$$

This is an integral equation in σ.
(Fredholm integral equation of the first kind).

The capacitance of the plate is defined as:

$$C = \frac{q}{V} = \frac{1}{V}\int_{-a}^{a} dx \int_{-a}^{a} dy\, \sigma(x,y) \quad (4)$$

Once we solve (3) for $\sigma(x,y)$ we can determine the potential anywhere in space and the capacitance of the plate.

Approximate Solution

break up the plate into sub-sections as shown in the figure, and define pulse Fcns,

$$f_n = \begin{cases} 1 & \text{on } \Delta S_n \\ 0 & \text{on all other } \Delta S_m \end{cases} \quad (5)$$

expand the charge density in terms of these "basis Fcns"

$$\sigma(x,y) \simeq \sum_{n=1}^{N} \alpha_n f_n \quad (6)$$

substitute this expansion into the integral equation (3):

$$V = \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{\sum_{n=1}^{N} \alpha_n f_n}{4\pi\epsilon\sqrt{(x-x')^2+(y-y')^2}} \quad (7)$$

now use point-matching, i.e. enforce this equation at the N centers of each sub-section

$$\therefore \quad V = \int_{-a}^{a}\int_{-a}^{a} \frac{\sum \alpha_n f_n}{4\pi\epsilon\sqrt{(x_m - x')^2 + (y_m - y')^2}}\, dx'\, dy' \qquad (8)$$

$$m = 1, \dots N$$

$$V = \sum_{n=1}^{N} \alpha_n l_{mn} \qquad m = 1, \dots, N$$

$$l_{mn} = \underbrace{\int_{\Delta x_n} dx' \int_{\Delta y_n} dy'}_{\substack{\text{surface}\\ \text{of each}\\ \text{subsection}}} \frac{1}{4\pi\epsilon\sqrt{(x_m - x')^2 + (y_m - y')^2}} \qquad (9)$$

$$[V] = [l_{mn}][\alpha_n] \qquad V_m = V$$

$$[\alpha_n] = [l_{mn}]^{-1}[V_m]$$

Thus l_{mn} represents the potential at the center of $\underline{\Delta S_m}$, due to a uniform charge density of unit amplitude over $\underline{\Delta S_n}$

The capacitance can then be calculated by substituting the expansion (6) into (4).

$$C = \frac{q}{V} \simeq \frac{1}{V} \int_{-a}^{a} dx \int_{-a}^{a} dy \sum_{n=1}^{N} \alpha_n f_n \approx \frac{1}{V} \sum_{n=1}^{N} \alpha_n \Delta S_n \qquad (10)$$

$$= \sum_{mn} l_{mn}^{-1} \Delta S_n$$

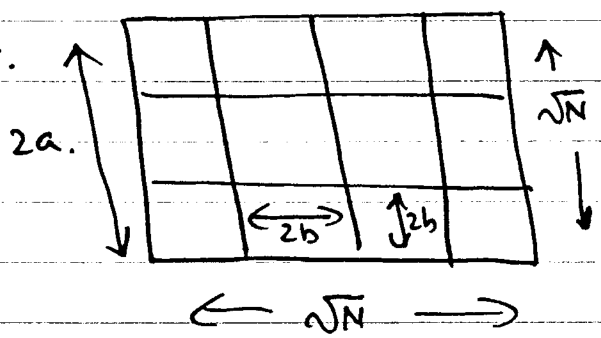
l_{mn}^{-1} is inverse of $[l_{mn}]$.

this says - capacitance of an object is the sum of the capacitances of all its subsections plus the mutual capacitances between every pair of subsections.

Numerical Results

in general the integral of eq. ⓐ will have to be evaluated to find l_{mn} and then $[l_{mn}]^{-1}$ evaluated.

let us use the same number of subsections in each direction → square grid

∴  N total subsection

$\therefore$ $2b = \dfrac{2a}{\sqrt{N}}$ is the length of the sides of ΔS_n.

if we assume, that instead of pulse fcns, the charge density is concentrated at the center of each sub-section

with $\sigma(x,y) = \delta(x_n, y_n)\,\Delta S_n$

then $$l_{mn} \approx \frac{\Delta S_n}{4\pi\epsilon R_{mn}} = \frac{b^2}{\pi\epsilon\sqrt{(x_m-x_n)^2+(y_m-y_n)^2}} \qquad (11)$$

the further away ΔS_n and ΔS_m are the more accurate this solution is.

with $N = 36$ the results using eq ⑪ give.

$$\frac{C}{2a} = 39.2 \ (\text{pF/meter})$$

the exact answer is 40 (pF/meter).

Two-dimensional Electromagnetic Fields

time harmonic — $e^{j\omega t}$ time variation

$$\bar{\nabla}\times\bar{E} = -j\omega\mu\bar{H}$$
$$\bar{\nabla}\times\bar{H} = j\omega\epsilon\bar{E}+\bar{J}$$

consider $\partial_z = 0$, TM fields, isotropic media, and homogeneous,

assume $\bar{\nabla}\times\bar{A} = \mu\bar{H} = \bar{B} \Rightarrow \bar{\nabla}\cdot\bar{B} = 0$

(i.e. magnetic flux is divergenceless).

$$\therefore \bar{\nabla}\times\bar{E} = -j\omega\bar{\nabla}\times\bar{A} \Rightarrow \bar{E} = -j\omega\bar{A}+\bar{\nabla}\phi$$

$$\bar{\nabla}\times\left(\frac{\bar{\nabla}\times\bar{A}}{\mu}\right) = j\omega\epsilon\bar{E}+\bar{J}$$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{A} = j\omega\mu\epsilon\bar{E}+\mu\bar{J} = \omega^2\mu\epsilon\bar{A}+j\omega\epsilon\mu\bar{\nabla}\phi+\mu\bar{J}$$
$$\equiv \bar{\nabla}(\bar{\nabla}\cdot\bar{A})-\bar{\nabla}^2\bar{A}$$

Lorentz - guage: $\bar{\nabla}\cdot A - j\omega\epsilon\mu\phi = 0$

$$\therefore \boxed{\bar{\nabla}^2\bar{A}+\omega^2\mu\epsilon\bar{A} = -\mu\bar{J}}$$ (Helmholtz equation)

$$\bar{E} = -j\omega\bar{A}+\bar{\nabla}\left(\frac{\bar{\nabla}\cdot\bar{A}}{j\omega\mu\epsilon}\right)$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) = (\partial_{xx}A_x+\partial_{xy}A_x+\partial_{xz}A_x)\,\hat{a}_x+(\partial_{yx}A_y+\partial_{yy}A_y+\partial_{yz}A_y)\,\hat{a}_y$$
$$+(\partial_{zx}A_z+\partial_{zy}A_z+\partial_{zz}A_z)$$

but $\partial_z \equiv 0$ $\quad\therefore\quad \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) = (\partial_{xx}A_x + \partial_{xy}A_x)\,\hat{a}_x + (\partial_{yx}A_y + \partial_{yy}A_y)\,\hat{a}_y$

if $\quad \bar{J} = J_z\,\hat{a}_z$

$$\bar{\nabla}^2 A_z + \omega^2\mu\epsilon A_z = -\mu J_z$$

(in isotropic media this means $\bar{E} = \hat{a}_z E_z$)

$$\bar{E} = \hat{a}_z E_z \doteq -j\omega A_z$$

$$\therefore \quad \nabla^2 E_z + \omega^2\mu\epsilon E_z = j\omega\mu J_z$$

let $k^2 = \omega^2\mu\epsilon$ $\qquad k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$ (wavenumber)

$$\therefore \quad \boxed{\nabla^2 E_z + k^2 E_z = j\omega\mu J_z}$$

the general solution can be obtained by first finding the Green's fcn for this problem, i.e. field due to a current filament of current I at $\bar{\rho}' = \hat{a}_x x' + \hat{a}_y y'$,

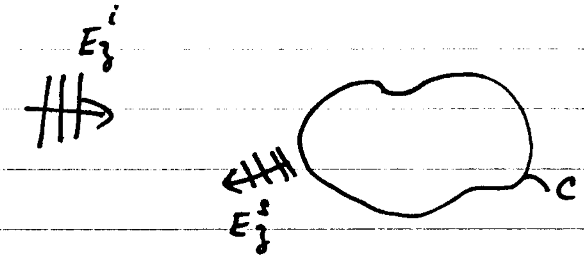
$$\boxed{E_z = -\frac{k\eta}{4} I \; H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)}$$

$$\eta = \sqrt{\mu/\epsilon} \approx 120\pi \approx 377\,\Omega$$

$$\therefore \quad \boxed{E_z(\bar{\rho}) = -\frac{k\eta}{4} \iint_{\substack{\text{source}\\ \text{region}}} J_z(\bar{\rho}')\, H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\, ds'}$$

Scattering from Conducting Cylinders, TM case.

Some incident field impinges on a two-dimensional cylinder



and produces a scattered field. By superposition the total electric field outside the cylinder is a sum of these two fields.

$$E_z^T = E_z^s + E_z^i$$

and on the surface of the cylinder the total electric field must be equal to zero:

$$E_z^T = E_z^s + E_z^i = 0 \quad \text{on } C$$

$$\therefore \text{ on } C \quad E_z^s = E_z^i$$

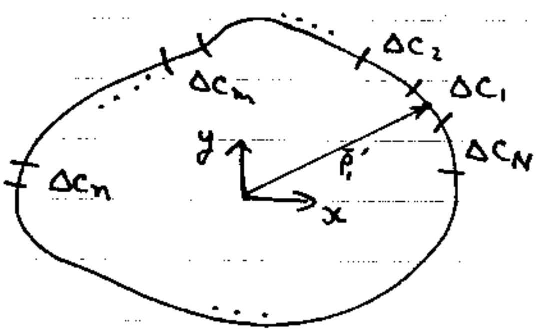
i.e. $$E_z^i(\bar{\rho}) = \frac{k\eta}{4} \int_C J_z(\bar{\rho}')\, H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\, dl' \qquad \bar{\rho} \text{ on } C$$

this is an integral equation in that we know $E_z^i(\bar{\rho})$ and are required to find $\bar{J}_z(\bar{\rho}')$

once we find $J_z(\bar{\rho}')$ (i.e. the surface current) then we can find the field anywhere outside the cylinder from the general solution.

we can now use the method of moments and expand the unknown $J_z(\bar{\rho}')$ in terms of pulse fcns:

$$f_n(\bar{\rho}) = \begin{cases} 1 & \text{on } \Delta C_n \\ 0 & \text{on all other } \Delta C_m \end{cases}$$



$$J_z = \sum_{n=1}^{N} \alpha_n f_n$$

substituting into the integral equation

$$E_z^i(\bar{\rho}) = \frac{k\eta}{4} \int_C \left(\sum_{n=1}^{N} \alpha_n f_n\right) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\, dl'$$

this represents one equation in the N unknowns. if we use point-matching for testing

$$E_z^i(\bar{\rho}_m) = \frac{k\eta}{4} \int_{\Delta C_n'} \sum \alpha_n H_0^{(2)}(k|\bar{\rho}_m - \bar{\rho}_n'|)\, dl \qquad m=1,\dots,N$$

or defining:

$$\begin{cases} l_{mn} = \dfrac{k\eta}{4} \displaystyle\int_{\Delta C_n} H_0^{(2)}\left[k\sqrt{(x-x_m)^2+(y-y_m)^2}\,\right] dl \\ g_m = E_z^i(x_m, y_m) \end{cases}$$

then we have the matrix equation:

$$[l_{mn}][\alpha_n] = [g_m]$$

$$\boxed{J_z = [f_n]^T [l_{nm}^{-1}][g_m]}$$

we now must determine the integral for l_{mn} ; this cannot be done analytically. In general, the element dl of ΔC_n follows the curve of the cylinder over ΔC_n.

The simplest approximation is to assume ΔC_n so small that the argument of the integral is constant:

$$l_{mn} \approx \frac{k\eta}{4} \Delta C_n \, H_0^{(2)}\left[k\sqrt{(x_n-x_m)^2+(y_n-y_m)^2}\,\right] \quad m \neq n$$

but when $m = n$ $H_0^{(2)}$ has a singularity which we must integrate analytically.

the small argument formula for the Hankel Fcn is

$$H_0^{(2)}(z) \approx 1 - j\frac{2}{\pi}\log\left(\frac{\gamma z}{2}\right) \qquad \gamma = 1.781\ldots$$

(Euler's constant)

$$\therefore\ l_{nn} = \frac{k\eta}{4}\int_{\Delta C_n}\left[1 - j\frac{2}{\pi}\log\left[\frac{\gamma k}{2}\sqrt{(x-x_m)^2+(y-y_m)^2}\right]\right]dl$$

$$\int \log x\, dx = x\log\left(\frac{x}{e}\right)$$

if we assume ΔC_n is a straight line then $\sqrt{(x-x_m)^2+(y-y_m)^2} = l$ = distance from midpoint of line

$$\int_{\Delta C_n}\log\left[\frac{\gamma k}{2}\sqrt{(x-x_m)^2+(y-y_m)^2}\right]dl$$

$$\approx \int_{-\frac{\Delta C_n}{2}}^{\frac{\Delta C_n}{2}}\log\left[\frac{\gamma k}{2}|l|\right]dl$$

$$= \left[\, l\log\left[\frac{\gamma k|l|}{2e}\right]\right]_{-\frac{\Delta C_n}{2}}^{\frac{\Delta C_n}{2}}$$

$$= \Delta C_n\log\left[\frac{\gamma k\,\Delta C_n}{4e}\right]$$

$$\therefore\ \boxed{l_{nn} \approx \frac{\eta}{4}k\,\Delta C_n\left[1 - j\frac{2}{\pi}\log\left(\frac{\gamma k\,\Delta C_n}{4e}\right)\right]}$$