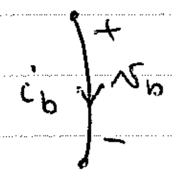
Graph Theoretic Formulation of Network Equations

(Vlach, Singhal, Ch. 3)



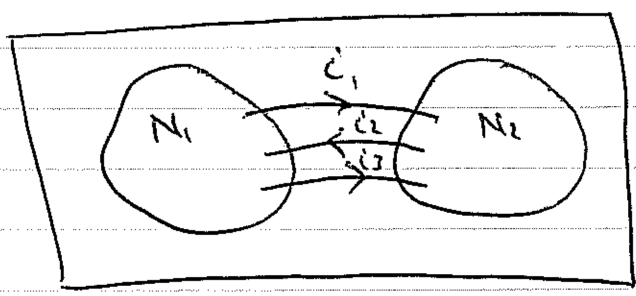
v_b is branch voltage

i_b is branch current

<u>Network</u> is made up of interconnected branches: b - branches $n+1$ nodes

KVL, KCL are written with respect to the <u>topology</u> of the network.

If we partition the network into two sets of nodes:

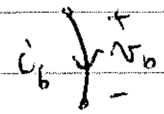


<u>Generalized KCL</u> says $i_1 + (-i_2) + i_3 = 0$

the partition is called a <u>cut</u>.

<u>KVL</u>: around any closed path the sum of voltage drops (or increases) is zero.

Note: if we're consistent and always use

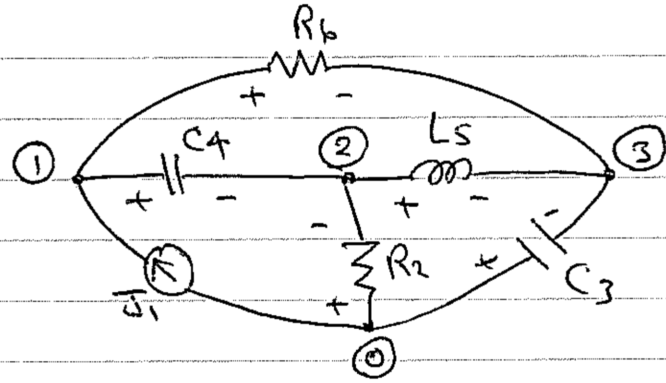


as the orientation, then we could just use

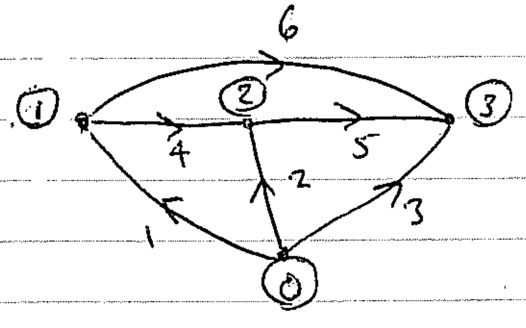
and agree that v_b drops in the direction of the arrow.

This will also give the passive power convention in which $i_b v_b$ represents the power absorbed by the branch.

Ex:



Graph of the network:



$n+1 = 4$

$b = 6$

(Node-Edge) Incidence Matrix

$A \in \mathbb{Z}^{n \times b}$ $\qquad A_{ij} = \begin{cases} +1 & \text{if edge } j \text{ leaves node } i \\ -1 & \text{if edge } j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases}$

For the above example:

$$A = \begin{array}{c|c} \text{node} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Note: node 0 is called the reference node and is not included in A.

It can be shown that rank $A = n$
i.e., the rows are independent

$$\boxed{\text{KCL: } A\underline{i} = 0}$$

where $\underline{i} = \begin{pmatrix} i_1 \\ \vdots \\ i_b \end{pmatrix}$ is vector of branch currents.

Introducing <u>nodal voltages</u>: v_n

v_{n1} is the voltage of node ① with respect to the reference node.

The branch voltages can be obtained, for e.g., as

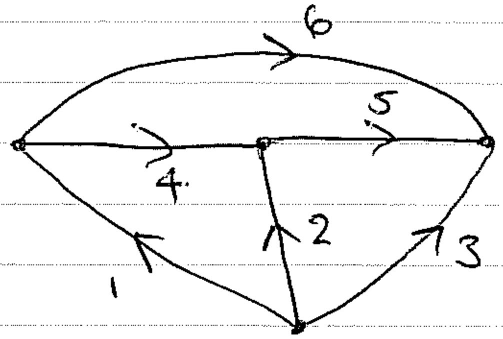
$$v_4 = v_{n1} - v_{n2}$$

or $\boxed{\underline{v} = A^T \underline{v_n}}$ a form of KVL.

where $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_b \end{pmatrix}$ $\underline{v_n} = \begin{pmatrix} v_{n1} \\ \vdots \\ v_{nn} \end{pmatrix}$

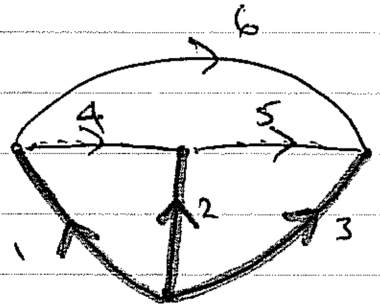
Cutset and Loopset Matrices

tree:



$n+1=4$
$b=6$

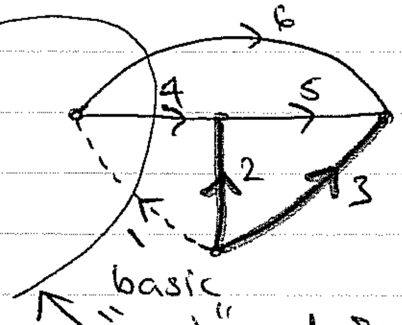
consider the tree made up of branches (edges) 1, 2, 3. (called twigs). (a tree will have n branches and no loops).



chords are the remaining branches (edges). 4, 5, 6.

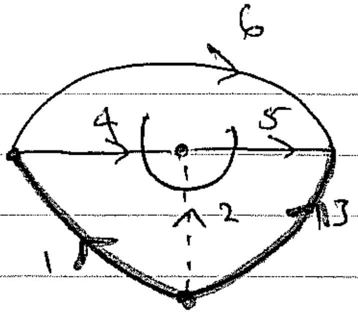
these define the cotree

if we remove one of the n twigs, we get a partition of nodes of the tree: e.g. remove 1

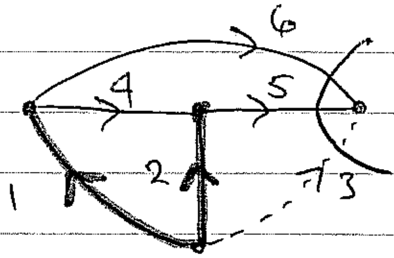


KCL: $i_1 - i_4 - i_6 = 0$

"cut" defined by removing one twig.



KCL: $i_2 + i_4 - i_5 = 0$



KCL. $i_3 + i_5 + i_6 = 0$

Note: the twig which was removed to define the cut determines the positive direction for the KCL.

The cutset matrix Q:

$$
Q = \begin{array}{c} \text{basic cuts} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \text{edges} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}
$$

$\boxed{Q\,\underline{i} = 0}$ KCL based on cut

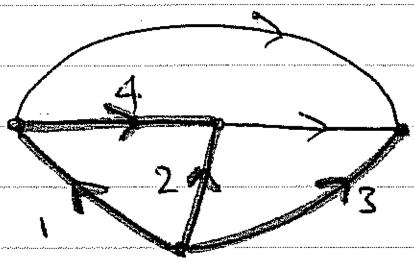
If we number our twigs as $1 \ldots n$

then the form of Q will always be

$$Q = [\,Q_t \quad Q_c\,] = [\,1 \quad Q_c\,] \quad (n \times b)$$

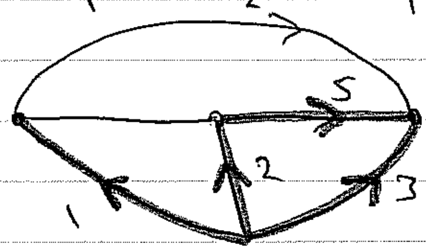
$Q_t = 1$ $n \times n$ identity matrix.

Now take the tree and add one chord at a time. (Note there are $b-n$ chords).

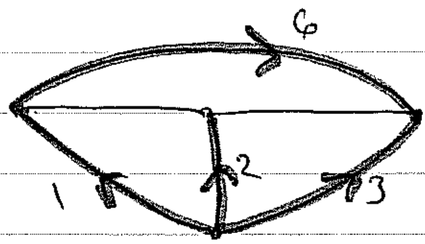


this defines a loop for which we can write KVL.

KVL: $v_1 - v_2 + v_4 = 0$



KVL: $v_2 - v_3 + v_5 = 0$



KVL: $v_1 - v_3 + v_6 = 0$

the <u>loopset matrix</u> B:

$$
B = \begin{array}{c} \text{loops} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} & & \text{edges} & & & \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{ccc:ccc}
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$B = [B_t \;\; B_c] = [B_t \;\; 1]. \quad (b-n) \times b$$

$B_c = 1 \quad b-n \times b-n$ identity matrix.

It can be shown that

$$\underset{(b-n)\times b}{B}\ \underset{b\times n}{Q^T} = 0 \qquad (b-n)\times n$$

and

$$\underset{(n\times b)}{Q}\ \underset{b\times(b-n)}{B^T} = 0 \qquad n\times(b-n)$$

i.e.: rows of B are <u>orthogonal</u> to rows of Q

In partitioned form:

$$B\,Q^T = \begin{bmatrix} B_t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ Q_c^T \end{bmatrix} = B_t + Q_c^T = 0$$

or $\boxed{B_t = -Q_c^T}$

So B can be written as $\boxed{B = \begin{bmatrix} -Q_c^T & 1 \end{bmatrix}}$

$$\boxed{Q = \begin{bmatrix} 1 & -B_t^T \end{bmatrix}}$$

Independent Variables

consider the cutset matrix KCL:

$$Q \underline{i} = 0$$

and partition $\underline{i} = \begin{pmatrix} \underline{i}_t \\ \underline{i}_c \end{pmatrix}$ $\quad \underline{i}_t$ twig currents, $\underline{i}_c$ chord currents.

$$Q\underline{i} = [1 \quad Q_c] \begin{pmatrix} \underline{i}_t \\ \underline{i}_c \end{pmatrix} = i_t + Q_c i_c = 0$$

$$\therefore \boxed{i_t = -Q_c i_c}$$

twig currenots can be obtained in terms of chord currents.

$$\therefore \underline{i} = \begin{pmatrix} \underline{i}_t \\ \underline{i}_c \end{pmatrix} = \begin{bmatrix} -Q_c \\ 1 \end{bmatrix} \underline{i}_c = \begin{bmatrix} B_t^T \\ 1 \end{bmatrix} \underline{i}_c$$

$$\boxed{\underline{i} = B^T \underline{i}_c}$$

$\therefore$ chord currents can be considered as independent variable.

Now consider KVL based on loopset matrix:

$$B\,\underline{v} = 0$$

$$\begin{bmatrix} B_t & 1 \end{bmatrix} \begin{pmatrix} v_t \\ v_c \end{pmatrix} = B_t v_t + v_c = 0 \implies v_c = -B_t v_t$$

$$v = \begin{pmatrix} v_t \\ v_c \end{pmatrix} = \begin{pmatrix} I \\ -B_t \end{pmatrix} v_t = \begin{pmatrix} 1 \\ Q_c^T \end{pmatrix} v_t$$

$$\boxed{\underline{v} = Q^T \underline{v}_t}$$

$\therefore$ <u>twig voltages</u> should be considered as independent variables.

For this reason we should put <u>independent current sources on the cotree</u> and <u>independent voltage sources on the tree</u>

<u>One way to incorporate Sources.</u>

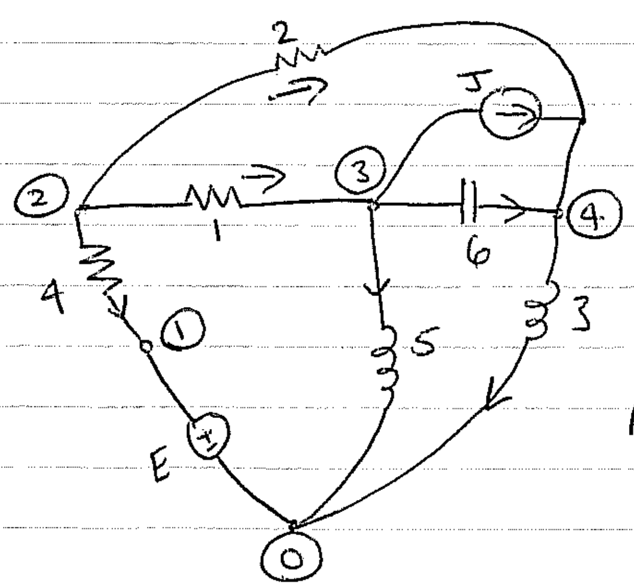
(E.g. of nodal analysis).

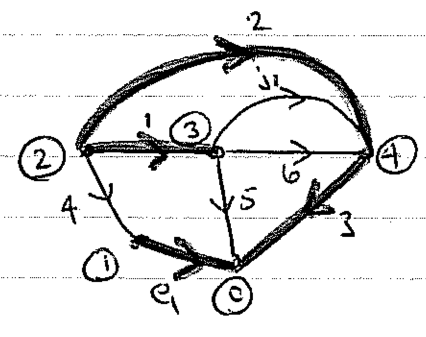
- put all independent current sources in cotree. (chords)
- put all independent voltage sources in tree (twigs).

1. start tree by labelling all voltage sources as $e_1, \ldots, e_\ell$
2. complete tree with remaining passive elements. numbering starting with 1.
3. continue numbering edges of chords starting with passive elements.
4. end with current sources in cotree: $j_1, \ldots, j_m$

Thus, produce the <u>augmented matrix</u>:

<u>Example:</u>





$$
A_a = \begin{array}{c} \\ \left[\begin{array}{c|cccccc|c} e_1 & 1 & 2 & 3 & 4 & 5 & 6 & j_1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 \end{array}\right] \end{array} \begin{array}{c} \text{node} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
$$

$$= [A_e \,\vdots\, A \,\vdots\, A_j]$$

$$
\begin{array}{c}
\text{edges} \\
\begin{array}{cccccccccc}
 & e_1 & 1 & 2 & 3 & 4 & 5 & 6 & j_1 & \\
Q_a = & \left[\begin{array}{c|cccccc|c}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{array}\right] & & & & & & & & \begin{array}{c} e_1 \\ 1 \\ 2 \\ 3 \end{array}
\end{array}
\end{array}
$$

Cutset determined by edges

preserve identity matrix.

Both matrices give us <u>KCL</u> $\left.\begin{array}{l} A_a \underline{i} = 0 \\ Q_a \underline{i} = 0 \end{array}\right\}$

For <u>nodal analysis</u>, assume only ind. current sources (if there are voltage sources, assume they can be converted to current src's using Thévenin-Norton tx.).

— also assume that every passive element has an admittance constitutive relation:

$$\boxed{Y_b \underline{v}_b = \underline{i}_b}$$

$\underline{v}_b$, $\underline{i}_b$ contain the branch voltages and currents in the passive branches

Then we can us A_a or Q_a as follows:

let $\underline{i} = \begin{pmatrix} \underline{i}_b \\ \underline{j} \end{pmatrix}$ $\underline{j}$ - vector of chord branches with ind. current src's.

$$A_a \underline{i} = 0 \quad (\text{KCL})$$

$$\begin{bmatrix} A & \vdots & A_j \end{bmatrix} \begin{bmatrix} \underline{i}_b \\ \underline{j} \end{bmatrix} = 0 \qquad A\,\underline{i}_b = -A_j\,\underline{j}$$

using $Y\underline{v}_b = \underline{i}_b$: $\qquad A\,Y_b\,\underline{v}_b = -A_j\,\underline{j}$

<u>KVL</u> is written as $\qquad \underbrace{\underline{v}}_{\text{all branch voltages}} = A_a^T \underbrace{\underline{v}_n}_{\text{nodal voltages}}$

or
$$\begin{pmatrix} \underline{v}_b \\ \underline{v}_j \end{pmatrix} = \begin{bmatrix} A^T \\ A_j^T \end{bmatrix} \underline{v}_n \Rightarrow \begin{array}{l} \underline{v}_b = A^T \underline{v}_n \\ \underline{v}_j = A_j^T \underline{v}_n \end{array}$$

$\therefore \quad A\,Y_b\,A^T\,\underline{v}_n = -A_j\,\underline{j}$

$$\boxed{\begin{cases} Y\,\underline{v}_n = \underline{j}_n \\ Y = A\,Y_b\,A^T \\ \underline{j}_n = -A_j\,\underline{j} \end{cases}}$$

Nodal analysis using topological formulation.

<u>using Q</u> :

$Q_a\,\underline{i} = 0 \quad$ (KCL) $\qquad Q_a = [Q \;\vdots\; Q_j]$

$Q\,\underline{i}_b = -Q_j\,\underline{j} \quad$ (partition)

$Q\,Y_b\,\underline{v}_b = -Q_j\,\underline{j} \quad$ (substitute constitutive relation).

$Q\,Y_b\,Q^T\underline{v}_t = -Q_j\,\underline{j} \quad$ (KVL in terms of twig voltages).

or independence of twig voltages.

$$\underline{\underline{\underline{v} = Q_a^T v_t}} \Rightarrow \begin{cases} \underline{v}_b = Q^T v_t \\ \underline{v}_j = Q_j^T v_t \end{cases}$$

Nodal analysis in terms of twig voltages.

$$\begin{cases} Y\,\underline{v}_j = \underline{j}_t \\ Y = Q\,Y_b\,Q^T \\ \underline{j}_t = -Q_j\,\underline{j} \end{cases}$$

For <u>loop analysis</u>, assume only ind. voltage sources and that every passive element has <u>impedance type</u> constitutive relation

$$\boxed{Z_b \, \underline{i}_b = \underline{v}_b}$$

$$B_a = \left[\, B_E \;\vdots\; B \,\right]$$

B_E part of B_a with ind. voltage sources on the twigs numbered first.

start with <u>KVL</u>: $B_a \underline{v} = 0$

partition $\underline{v} = \begin{pmatrix} \underline{v}_E \\ \underline{v}_b \end{pmatrix}$ $\underline{v}_b$ – passive branches

$$\therefore \quad B_E \underline{v}_E + B \underline{v}_b = 0 \implies B \underline{v}_b = -B_E \underline{v}_E$$

substitute constitutive relation:

$$B Z_b \underline{i}_b = -B_E \underline{v}_E$$

KCL: $\underline{i} = B_a^T \underline{i}_c$ ("KCL" in terms of chord currents) or independence of chord currents)

$$\underline{i} = \begin{pmatrix} \underline{i}_E \\ \underline{i}_b \end{pmatrix} = \begin{pmatrix} B_E^T \\ B^T \end{pmatrix} \underline{i}_c$$

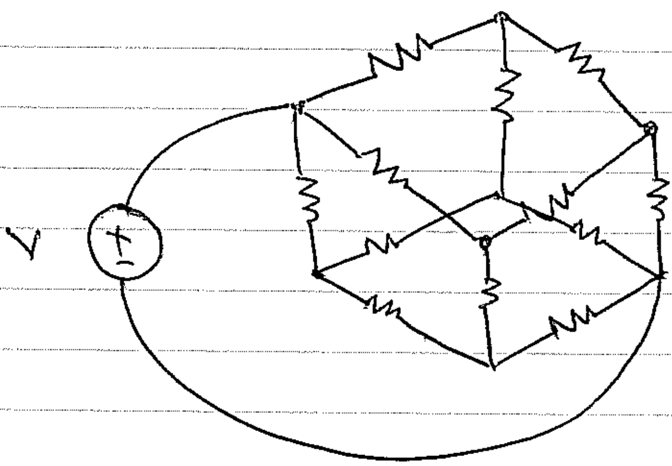
$$\underline{i}_E = B_E^T \underline{i}_c$$
$$\underline{i}_b = B^T \underline{i}_c$$

Substituting for $\underline{i}_b$:

$$B Z_b B^T \underline{i}_c = - B_E \underline{v}_E$$

$$\boxed{\begin{cases} Z \underline{i}_c = \underline{e}_\ell \\ Z = B Z_b B^T & \text{impedance matrix} \\ \underline{e}_\ell = - B_E \underline{v}_E & \text{loop voltages.} \end{cases}}$$ Loop Analysis

Try formulating the ("all") Loop analysis for the following non-planar circuit:



All resistors have value R Ω.

Find the impedance seen by the voltage source.